Math 2210 - Section 12.4 Notes

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1 Differentiability

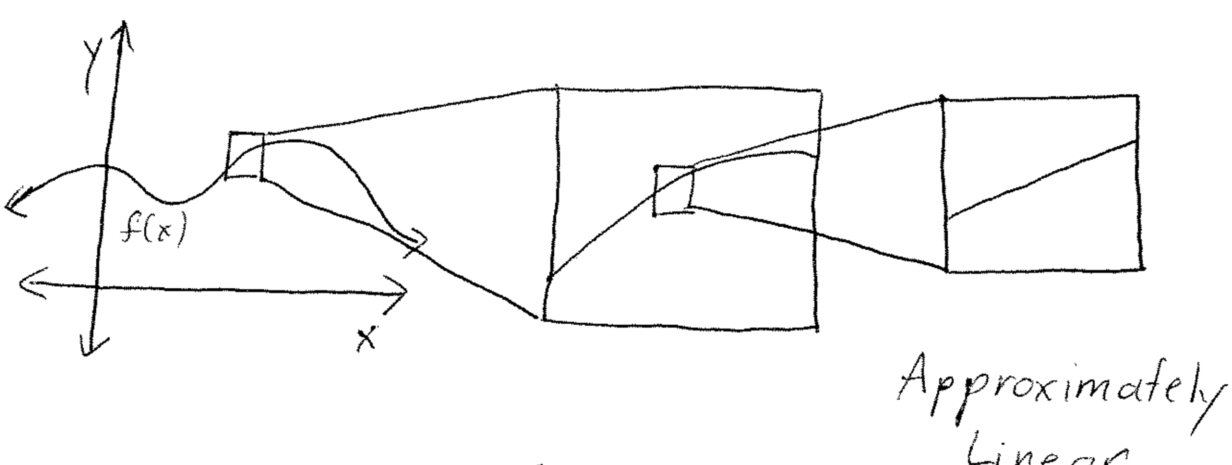
1.1 Tangent Lines and Tangent Planes

1.1.1 Single Variable Calculus

For a function of a single variable, we defined differentiability of the function f at the point x to mean the existance of the derivative $f'(x)$ at that point. For a single-variable function this meant that the limit:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

had to exist. We also saw that existance of the derivative at the point x meant that, if we zoomed in on the function around that input value we would see that the function looked more and more like a straight line.



The derivative of a point tells us the slope of the tangent line at that point, and a function being differentiable at a point means this makes sense. In other words, as we zoom in closer and closer, the function becomes more and more linear.

1.1.2 Multivariable Calculus

Now, if we want to extend our definition to a function of two variables, we may be tempted to define the derivative at a point (x, y) to be the limit:

$$\lim_{\mathbf{h}\to 0} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})}{\mathbf{h}}$$

where here we are representing a point (x, y) by the position vector $\mathbf{x}$ that points from the origin to that point. However, unfortunately, this definition doesn't work, or even make sense, because we cannot make sense of vector division.

However, the other concept, that of approximating a single variable functions with a line around a point, does make sense for multivariable functions, although instead of an approximating line, we must use an approximating plane. Along these lines we get the following definition:

Definition - The multivariable function $f(x, y)$ is locally linear at the point (a, b) if

$$f(a+h_1, b+h_2) = f(a, b) + h_1 f_x(a, b) + h_2 f_y(a, b) + h_1\epsilon_1(h_1, h_2) + h_2\epsilon_2(h_1, h_2)$$

where $\epsilon_1(h_1, h_2) \to 0$ and $\epsilon_2(h_1, h_2) \to 0$ as $(h_1, h_2) \to 0$. Where $(h_1, h_2) \to 0$ refers to the vector going to zero with respect to its length.

1.2 Differentiability

1.2.1 Definition

We define differentiability at a point to be synonymous with local linearity.

Definition - The function f is *differentiable* at $\mathbf{p}$ if it is locally linear at $\mathbf{p}$. The function f is differentaible on an open set R if it is differentiable at every point in R.

The vector $(f_x(\mathbf{p}), f_y(\mathbf{p})) = f_x(\mathbf{p})\mathbf{i} + f_y(\mathbf{p})\mathbf{j}$ is denoted $\nabla f(\mathbf{p})$ and is called the *gradient* of f. So, using this terminology a function f is differentiable at $\mathbf{p}$ if and only if:

$$f(\mathbf{p}+\mathbf{h}) = f(\mathbf{p}) + \nabla f(\mathbf{p}) \cdot \mathbf{h} + \epsilon(\mathbf{h}) \cdot \mathbf{h}$$

where $\epsilon(\mathbf{h}) \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.

We note that with this formulation the definition of differentiability extends naturally to any number of dimensions.

Example - Find the gradient of the function:

$$f(x, y) = x^3y - y^3$$

$$f_x(x,y) = 3x^2y$$

$$f_y(x,y) = x^3 - 3y^2$$

$$\nabla f = \langle 3x^2y, x^3 - 3y^2 \rangle$$

Example - Find the gradient of the function:

$$f(x, y) = (\sin(x^2y))^3$$

$$f_x(x,y) = 3\sin^2(x^2y)\cos(x^2y)(2xy)$$

$$= 6xy\sin^2(x^2y)\cos(x^2y)$$

$$f_y(x,y) = 3x^2\sin^2(x^2y)\cos(x^2y)$$

$$\nabla f = \langle 6xy\sin^2(x^2y)\cos(x^2y), 3x^2\sin^2(x^2y)\cos(x^2y) \rangle$$

1.2.2 Major Theorem

Now, checking that a function $f(x, y)$ is differentiable at a point may appear hard, but it's in fact pretty easy, and only involves in almost all cases examining the partial derivatives at the point. The following theorem is proven in the textbook, although we won't have time to go over the proof in class.

Theorem - If $f(x, y)$ has continuous partial derivatives $f_x(x, y)$ and $f_y(x, y)$ on a disk D whose interior contains (a, b) then $f(x, y)$ is differentiable at (a, b).

1.3 Tangent Planes

If the function f is differentiable at $\mathbf{p}_0$, then, when $\mathbf{h}$ has small magnitude

$$f(\mathbf{p}_0 + \mathbf{h}) \approx f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot \mathbf{h}.$$

If we let $\mathbf{p} = \mathbf{p}_0 + \mathbf{h}$ we can define a function T by:

$$T(\mathbf{p}) = f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)$$

and we see that $T(\mathbf{p})$ should be a good approximation to $f(\mathbf{p})$ if $\mathbf{p}$ is close to $\mathbf{p}_0$. The function $T(\mathbf{p})$ defines a plane, called the tangent plane at $\mathbf{p}$.

Example - For the function $f(x, y) = x^3y + 3xy^2$, find the equation of the tangent plane at $\mathbf{p}_0 = (2, -2)$.

$$z = 2^3(-2) + 3(2)(-2)^2 + \nabla f(2,-2) \cdot (\langle x,y\rangle - (2,-2))$$

$$= 8 + \ldots$$

$$f_x(2,-2) = 3x^2y + 3y^2\Big|_{(2,-2)} = 3(2^2)(-2) + 3(-2)^2 = -12$$

$$f_y(2,-2) = x^3 + 6xy\Big|_{(2,-2)} = 2^3 + 6(2)(-2) = -16$$

$$\Rightarrow z = 8 + \langle -12, -16\rangle \cdot \langle x-2, y+2\rangle = 8 - 12x + 24 - 16y - 32$$

$$\Rightarrow \boxed{12x + 16y + z = 0}$$

1.4 Rules for Gradients

There are a few rules for handling gradients that follow directly from our rules for partial derivatives. These are:

1. $\nabla[f(\mathbf{p}) + g(\mathbf{p})] = \nabla f(\mathbf{p}) + \nabla g(\mathbf{p})$
2. $\nabla[\alpha f(\mathbf{p})] = \alpha \nabla f(\mathbf{p})$
3. $\nabla[f(\mathbf{p})g(\mathbf{p})] = f(\mathbf{p}) \nabla g(\mathbf{p}) + g(\mathbf{p}) \nabla f(\mathbf{p})$

1.5 Continuity and Differentiability

Just as with single variable functions, we know that a function $f(x, y)$ is differentiable at a point (a, b) only if the function is continuous there. It does not go the other way, however. A function may be continuous at a point, but not differentiable.