Math 2210 - Section 12.3 Notes

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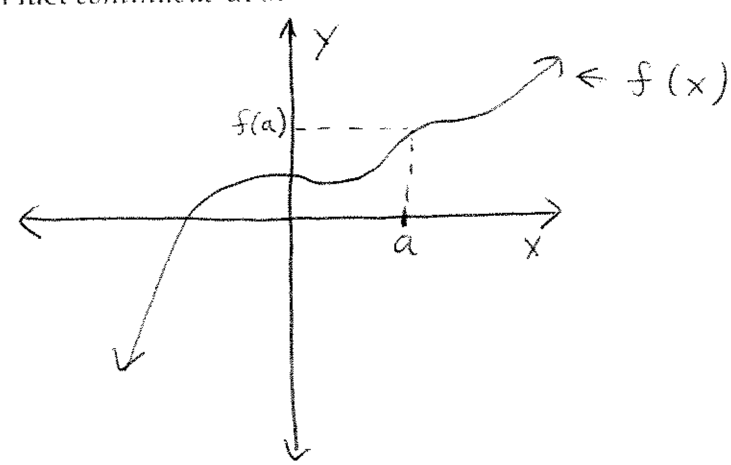
1 Limits and Continuity

1.1 Limits

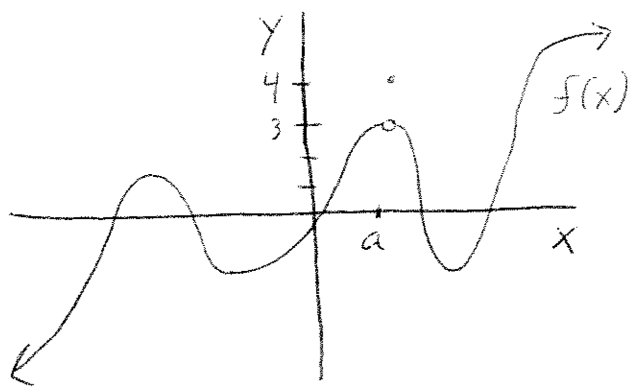
1.1.1 Single Variable Calculus

In calculus 1 we learned about limits and continuity in the context of functions of a single variable, $y = f(x)$. Basically, the idea is that the limit as a function $f(x)$ approaches an input value a is the value that the output of the function gets closer and closer to as the input of the function gets closer and closer to a.

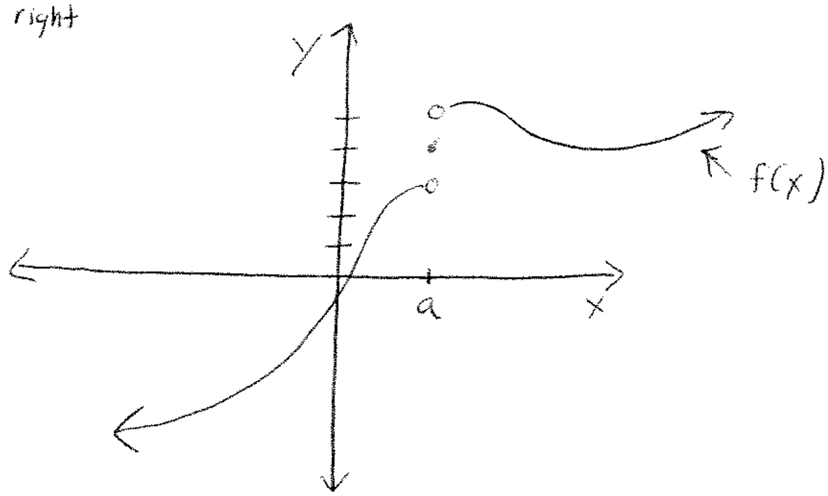
So, for example, in the drawing below we would have $\lim_{x \to a} = f(a)$, which would mean that not only does the limit exist at a, but that the function is in fact *continuous* at a.



On the other hand, for the function below we would have $\lim_{x\to a} f(x) = 3$, while $f(a) = 4$, and so the limit is well defined (as x gets really, really close to a, $f(x)$ gets really really close to 3) we find that the limit as the function appraoches a is different that the value of the function at a. This illustrates an important point, in that the limit is just concerned with the behavior of the function close to the limit point, not with the behavior of the function *at* the limit point, which may not even be well defined (as in the case of the derivative). However, in this example, while the limit is well defined, the function is not continuous at the point a.



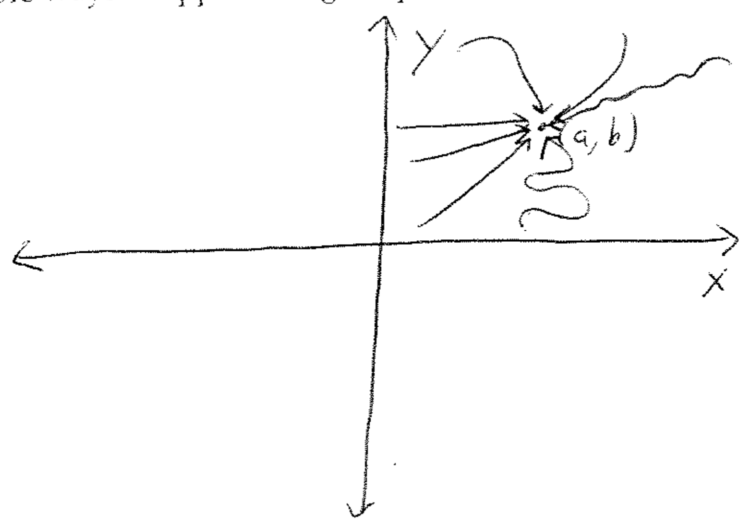
Finally, for the example below we would have that the limit as x approaches a from the ~~right~~ left, $\lim_{x\to a^+} f(x)$, is 3, while the limit as x approaches a from the ~~left~~ right, $\lim_{x\to a^-} f(x)$, is 5.



So, we would say that the limit of this function as x appraoches a is *undefined*, because we get different values depending on the direction of appraoch. Note that, if you remember, there are many ways that a limit may be undefined, and this is just one of those ways, although it's an important way to keep in mind as we discuss limits of multivariable functions.

1.1.2 Multivariable Functions and Formal Definitions

For multivariable functions $f(x, y)$ (we'll just be discussing functions of two variables here, although these ideas extend very naturally to functions of more than two variables) around a point (a, b) there are many, many possible ways of approaching the point in the xy-plane.



So, unlike with functions of a single variable, where we only had to worry about two possible directions of approach, in multivariable functions we have to worry about many, many (in fact, uncoutably infinite in most cases) possible curves of approach. This is what makes multivariable limits more difficult than single variable limits.

Now, for single variable functions you probably remember there was a nasty but necessary formal definition of limits involving epsilons and deltas. Well, there's no getting around it, and we have essentially the same definition for multivariable functions.

Definition

We say that

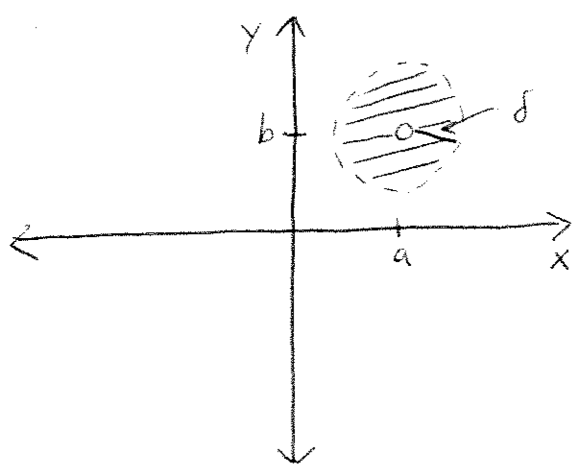
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that for each $\epsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x,y) - L| < \epsilon$, provided that $0 < ||(x,y) - (a,b)|| < \delta$.

Well, this looks almost exactly like our definition from single variable calculus. However, the difference comes from the term $||(x,y) - (a,b)||$, which means:

$$||(x,y) - (a,b)|| = \sqrt{(x-a)^2 + (y-b)^2}$$

the distance between the point (a,b) and the point (x,y). Now, if we say that $0 < ||(x,y) - (a,b)|| < \delta$, we're dealing with a whole punctured open disk around the point (a,b), and, as we said before, within this disk there are many, many possible ways of appraoching the point (a,b).



Some important things to keep in mind about this definition are:

1. The path of appraoch to (a, b) is irrelevant. So, if we take diferent paths and get different limiting values, then our limit does not exist.

2. The behavior of $f(x, y)$ and the point (a, b) doesn't matter, and in fact $f(x, y)$ may not even be defined at (a, b). This is the same as with single variable functions and limits.

3. The definitions is phrased so that it immediately extends to functions of three or more variables, using our generalized notion of the distance between two points in any number of dimensions.

Proving a function has a limit at a given point (a, b) can be a hard thing to prove, as we'll see in some examples later on. However, we will first state some facts about some common functions. Note we will not be proving these facts (although the proofs aren't hard), we'll just be taking them as true.

1.1.3 Limits of Common Simple Functions

Theorem

If $f(x, y)$ is a polynomial, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

and if $f(x, y) = p(x, y)/q(x, y)$, where p and q are polynomials (so $f(x, y)$ would be something called a *rational functions*), then

$$\lim_{(x,y)\to(a,b)} f(x, y) = \frac{p(a, b)}{q(a, b)}$$

provided $q(a, b) \neq 0$, naturally. Furthermore, if

$$\lim_{(x,y)\to(a,b)} p(x, y) = L \neq 0 \text{ and } \lim_{(x,y)\to(a,b)} q(x, y) = 0$$

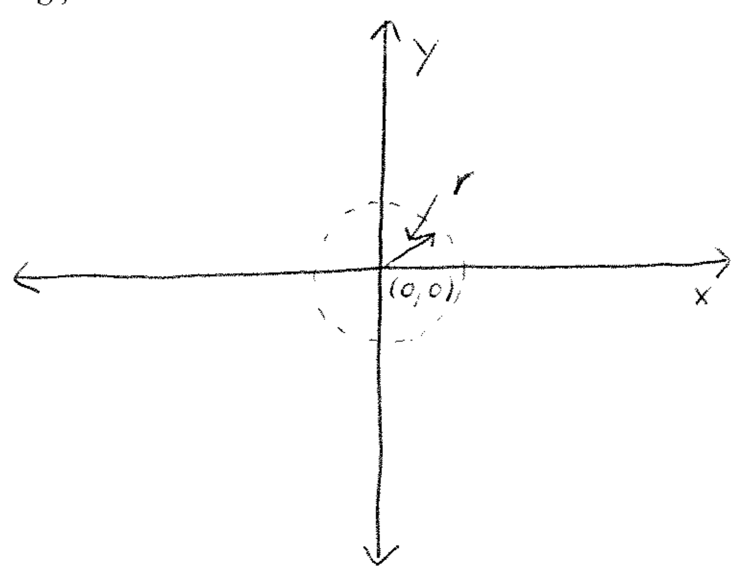
then

$$\lim_{(x,y)\to(a,b)} \frac{p(x,y)}{q(x,y)}$$

does not exist.

Example
Find $\displaystyle\lim_{(x,y)\to(-2,1)} (xy^3 - xy + 3y^2)$

Now, it is often easier to analyze limits of functions of two variables, especially limits at the origin, by changing to polar coordinates. The important point is that $(x,y) \to (0,0)$ if and only if $r = \sqrt{x^2+y^2} \to 0$. Thus, limits for functions of two variables can sometimes be expressed as limits involving just the one variable r.



Example

Find $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$

Example

Show that $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$ does not exist.

1.2 Continuity

1.2.1 Continuity at a Point

A function of two variables, $f(x, y)$, is called *continuous* at a point (a, b) if:

1. The functions $f(x, y)$ is well defined at the point (a, b). For our course, well defined will mean that the function must have an output at that point that is a real number.

2. The limit:

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

   exists.

3. We have the equality:

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We note that this is essentially the same as the definition of continuity from single variable calculus.

We note, again without proof, that all polynomial functions are continuous everywhere, and that all rational functions are continuous everywhere the denominator is not 0.

A very useful way to prove a given function is continuous is to show that the function is a composition of continuous functions. The following theorem then guarantees that the composition function will be continuous as well.

Theorem

If a function g of two variables is continuous at the point (a, b) and a function f of one variable is continuout at $g(a, b)$, then the composition $f(g(x, y))$ is continuous at (a, b).

Again, we won't be proving this (that's for a more advanced class) but we will be using this to prove other functions are continuous.

Example
Show that $f(x, y) = \sin(x^3 - 4xy)$ is continuous everywhere.

Example
Determine where the function $f(x, y) = \ln(1 - x^2 - y^2)$ is continuous.

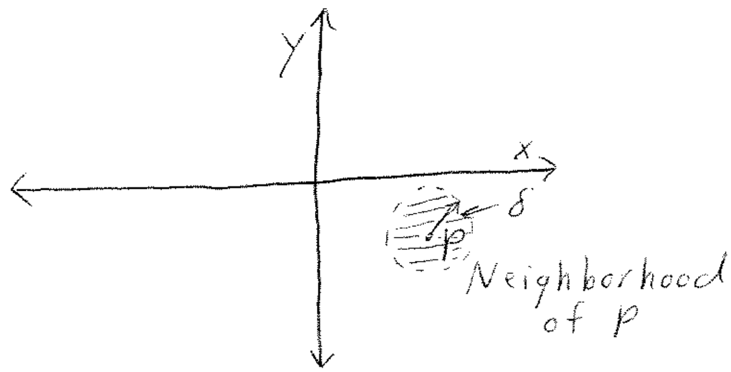
1.2.2 Continuity on a Set

We say that a function $f(x, y)$ is continuous on a set of points S if $f(x, y)$ is continuous at every point in that set. However, while this definition may seem simple (and in practice it usually is) there are a few subtleties.

First, we want to introduce some concepts from what is known in math as "metric topology", which is a fancy term that you'll deal with much, much more if you take an analysis class, but here we'll just be talking about some very basic ideas behind it. We also note that all the sets we talk about here will be sets of points in some finite dimensional space with the standard Euclidean distance. If this sounds a bit complicated, don't worry, it just means we'll be dealing with sets of points like we've been dealing with exclusively up to this point.

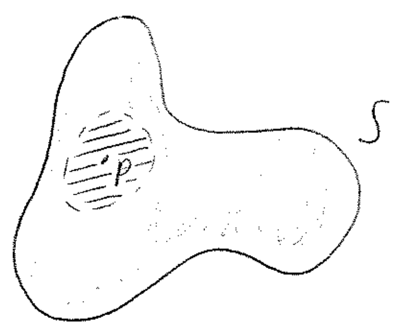
Definition

A *neighborhood* of radius δ of a point P is the set of all points Q such that $||Q - P|| < \delta$. In other words, all points a distance less than δ from the point P.



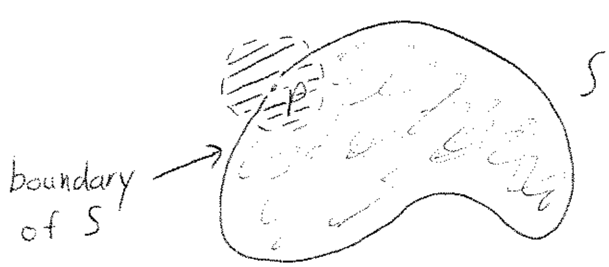
Definition

We say that a point P in a set S is an *interior point* if there is a neighborhood of the point P that is completely contained in S.



Definition

A point B is a *boundary point* of a set S if every neighborhood of B contains points that are in S and points that are not in S. The set of all boundary points of a set S is called the *boundary* of S.



Definition

A set is *open* if all its points are interior points. A set is *closed* if it contains all its boundary points. You can also define a set as being open if it is the complement of a closed set.

Phew! That's a lot of definitions, but none of them are that difficult. The important thing about all of these definitions is that it allows us to state precisely what we discussed earlier about equalify of mixed partial derivatives.

Theorem

If f_{xy} and f_{yx} exist and are continuous on an open set S, then $f_{xy} = f_{yx}$ at each point of the set S.

Example

For the set $\{(x, y) : x^2 + y^2 < 4\}$, first sketch the set, then describe the set's boundary. Is the set open, closed, or neither?