Math 2210 - Section 12.2 Notes

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1 Partial Derivatives

1.1 Definitions

In single variable calculus we recall that the derivative of a function $f(x)$ at a point $a \in \mathbb{R}$ is defined to be:

$$f'(a) = \lim_{h \to a} \frac{f(a+h) - f(a)}{h}$$

Or, in general, the derivative is a function of x in its own right and is defined as:

$$f'(x) = \lim_{h \to a} \frac{f(x+h) - f(x)}{h}$$

With multivariable functions everything becomes a little bit more difficult, and in no situation is this more clear than in the case of limits, but our basic approach remains the same.

For a function $f(x, y)$ of two variables we define the *partial derivative* of the function with respect to the variable x to be:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

while the partial derivative with respect to the variable y is:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Now, in practice this isn't very difficult at all. If we're asked, for example, to take the partial derivative of a function with respect to x we just treat y and all other variables as if they were constants.

Example
For the function $f(x, y) = e^x \cos y$ calculate $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = e^x \cos y$$

$$f_y(x, y) = -e^x \sin y$$

See, not so difficult at all. If you can take a derivative, you can take a partial derivative. Here's another example.

Example
For the function $f(x, y) = x \cos xy$ calculate $f_x(x, y)$ and $f_y(x, y)$.

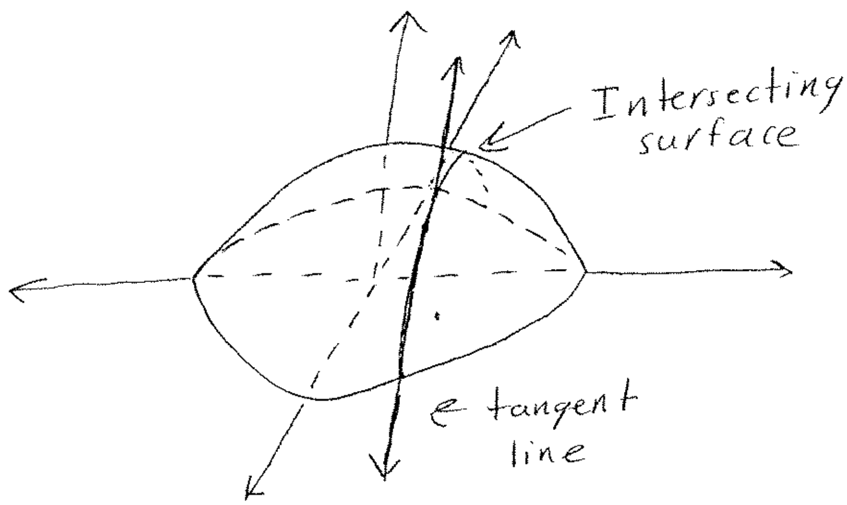
$$f_x(x, y) = \cos(xy) - xy \sin(xy)$$

$$f_y(x, y) = -x^2 \sin(xy)$$

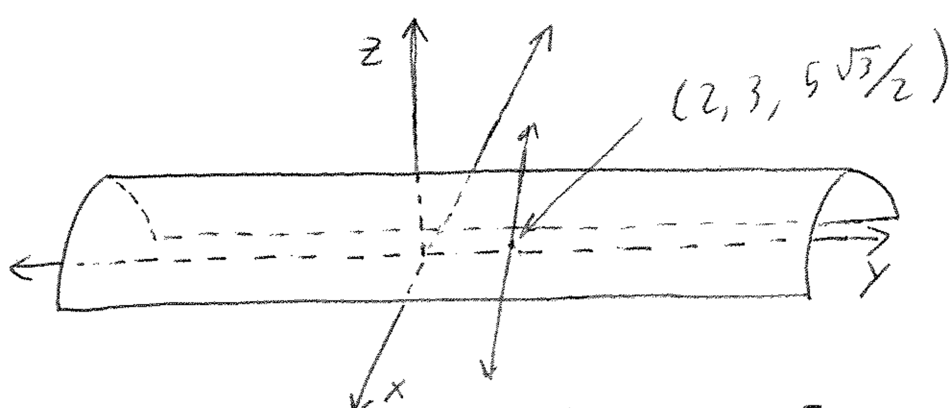
1.2 Geometric Interpretation

We can view the function $f(x, y)$ as a surface, $z = f(x, y)$. For a point (a, b) in the domain D of $f(x, y)$ we can view the partial derivative with respect to x at the point (a, b) by intersecting the surface $z = f(x, y)$ with the plane $y = b$.

The intersection of the surface $z = f(x, y)$ with the plane $y = b$ gives us a curve that we can parameterize by its x-coordinate. The slope of the curve when $x = a$ is the partial derivative, $f_x(x, y)$ at the point (a, b).



Find the slope of the tangent to the curve of intersection of the cylinder $4z = 5\sqrt{16 - x^2}$ and the plane $y = 3$ at the point $(2, 3, 5\sqrt{3}/2)$.



$$z = \frac{5}{4}\sqrt{16 - x^2} \qquad \frac{\partial z}{\partial x} = \frac{-5x}{4\sqrt{16 - x^2}}$$

$$\frac{\partial z}{\partial x}(2,3) = \frac{-10}{4\sqrt{16-4}} = \boxed{-\frac{5}{4\sqrt{3}}}$$

1.3 Higher Order Partials

The partial derivative of a function $f(x, y)$ with respect to either variable is itself a multivariable function. For example, if $f(x, y) = x^2y + 2\sin x$ then $f_x(x, y) = 2xy + 2\cos x$ is also a multivariable function. There's no reason why we cannot then take partial derivatives of this newly created function. This is the second order partial derivative, and there are four of them for a 2 variable function, namely f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

Example

For the function $f(x, y) = x\cos xy$ calculate f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

$$f_{xx} = -y\sin(xy) - xy^2\cos(xy) - y\sin(xy)$$
$$= \boxed{-xy^2\cos(xy) - 2y\sin(xy)}$$
$$f_{xy} = -x\sin(xy) - x^2y\cos(xy) - x\sin(xy)$$
$$= \boxed{-x^2y\cos(xy) - 2x\sin(xy)}$$
$$f_{yy} = \boxed{-x^3\cos(xy)}$$
$$f_{yx} = \boxed{-x^2y\cos(xy) - 2x\sin(xy)}$$

Note here that $f_{xy} = f_{yx}$. This is equality of mixed partial derivatives, and it's almost always the case, especially for the functions with which we'll be dealing. Formally, we have equality of mixed partial derivatives in the neighborhood of a point if the first-order partial derivatives exist and are continuous in that neighborhood. We won't be proving this, but it's worth remarking upon.

1.4 Notation

There are two ways we will commonly use to represent partial derivatives; the way we've used so far, and another way that uses the partial derivative operator ∂.

We can represent the partial derivative of $f(x, y)$ with respect to x either as $f_x(x, y)$ or as $\frac{\partial f}{\partial x}$. Similarly for mixed partials we have:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Note there's some potential for confusion here, because, for example, if you're taking a mixed partial first with respect to x and then with respect to y you've got the second situation enumerated above, where the order of the variables reads left to right on the left hand side, but right to left in the denominator of the right hand side. What a pain! Fortunately, as I said, we almost always have equality of mixed partials, and so this potential confusion rarely matters, but it's good to know about.