Math 2210 - Section 12.1 Notes

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1 Functions of Two or More Variables

1.1 Basic Concepts and Terms

So far in calculus we've dealt primarily with functions of one variable, either the standard function from one variable to one variable, $y = f(x)$, or vector valued functions from one variable to many variables, $\mathbf{y} = \mathbf{F}(t)$. We've also dealt with surfaces, which can be viewed (at least locally) as functions from two variables to one. In this chapter we examine these types of functions (functions of two or more variables) in much greater detail, and we generalize the concepts of differential calculus to dealing with functions of this type.

A real-valuced function of two real variables is a map that assigns to each ordered pair (x, y) in some set D of the plane a unique real number $f(x, y)$. One example would be:

$$f(x, y) = x^2 + 3y^2.$$

This set D is called the *domain* of the function. If it is not specified, we take D to be all inputs for which the function makes sense. The *range* of a function is its set of possible output values. We call x and y the independent variables, and z the dependent variable.

We can graph this functions by associating the z value above any input (x, y) to be the output $f(x, y)$. So, we'd write this graph as $z = f(x, y)$. Graphs of this type are usually surfaces.

These ideas extend naturally to real-valued functions of three or more variables.

Example

For the function $f(x, y) = \frac{y}{x} + xy$ find $f(1, 2)$. What is the domain of $f(x, y)$?

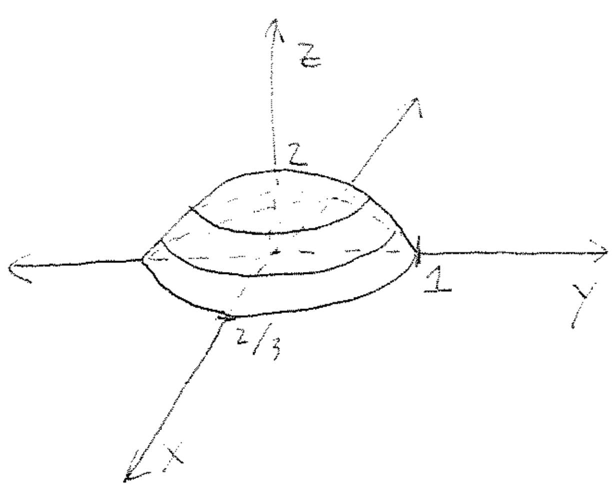
$$f(1,2) = \frac{2}{1} + (1)(2) = 4$$

Domain $\mathbb{R}^2 \setminus (0, y)$

All of $\mathbb{R}^2$ outside the y-axis.

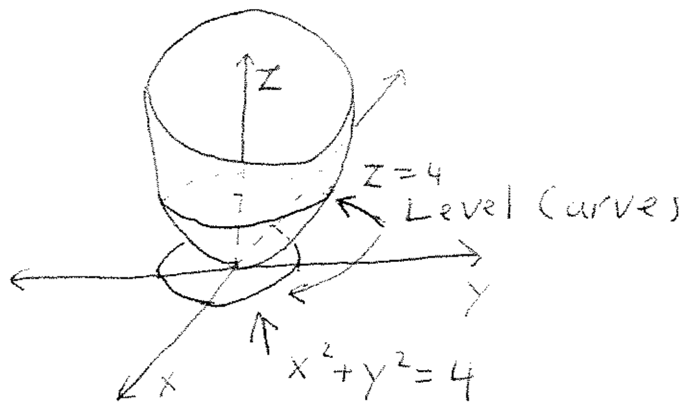
Example

Sketch the graph of $f(x, y) = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$.



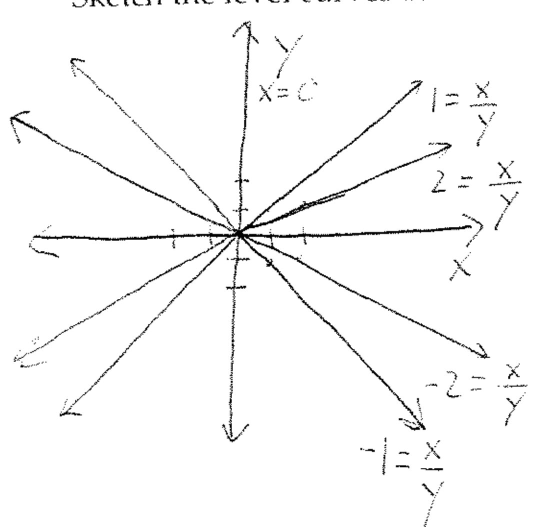
1.2 Level Curves

A level curve of a surface $z = f(x, y)$ is the curve defined by setting z to be a constant value. So, for example, if our function is $f(x, y) = x^2 + y^2$ a level curve would be all the points (x, y) for which $4 = x^2 + y^2$. We can represent this curve by either projecting it onto the xy plane, or by drawing it directly upon the surface. A collection of such curves is called a contour plot or contour map.



Example

Sketch the level curves at $z = -2, -1, 0, 1, 2$ for $z = \dfrac{x}{y}$.



$$-2 = \frac{x}{y} \Rightarrow y = -\frac{1}{2}x$$

$$-1 = \frac{x}{y} \Rightarrow y = -x$$

$$0 = \frac{x}{y} \Rightarrow x = 0$$

$$1 = \frac{x}{y} \Rightarrow y = x$$

$$2 = \frac{x}{y} \Rightarrow y = \frac{1}{2}x$$

Now, you can use a computer package to help you visualize these graphs. Maple and Mathematics are both good and are available in the computer lab.

1.3 Three or More Variables

These concepts extend naturally to functions of three or more variables. Functions of three variables occur all the time in nature. For example, if we use a three dimensional coordinate system to describe points inside the classroom, a three variable function would be the temperature at each point. Level surfaces (surfaces of constant temperature) would be called isotherms.

Example

Find the domain of

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$$

Defined when $x^2+y^2+z^2-9 \geq 0$

$\Rightarrow x^2+y^2+z^2 \geq 9$

So, outside the open ball of radius 3 centered at the origin.