Math 2210 - Section 11.9 Notes

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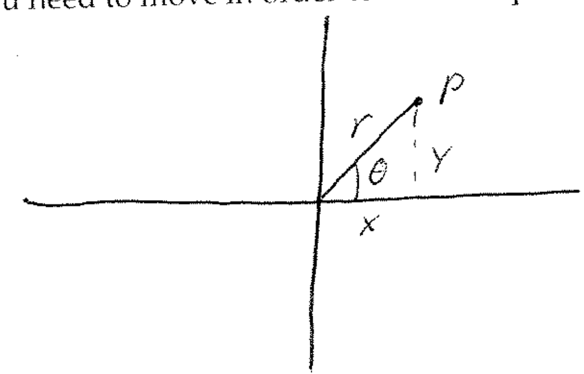
1 Cylindrical and Spherical Coordinates

1.1 Rectangular and Polar Coordinates

When we're dealing with the familiar two-dimensional coordinate plane, $\mathbb{R}^2$, we typically describe a point by specifying its x and y coordinates. That is, how far over in the x direction, plus how far over in the y direction, would you need to move in order to be at the point.



Now, as most of you have probably seen in previous math classes, we can also specify a point in $\mathbb{R}^2$ using a different type of coordinate system, namely, polar coordinates. When using polar coordinates we draw a line segment from the origin to the given point, and then specify the length of the line segment, r, and the angle this segment makes with the positive x-axis, θ. We note that the domain of r is $[0, \infty)$, while the range of θ is $[0, 2\pi)$. We also note that if $r = 0$ then all values of θ specify the same point, namely, the origin, and so there is a level of ambiguity in polar coordinates that is not present in rectangular (Cartesian) coordinates.

A question that naturally arises is, given a point with a representation in one system (either Cartesian or polar), how can we translate that into a representation in another coordinate system? Well, the formulas for doing this can be derived from simple geometry (try it!) and are:

Cartesian to Polar

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

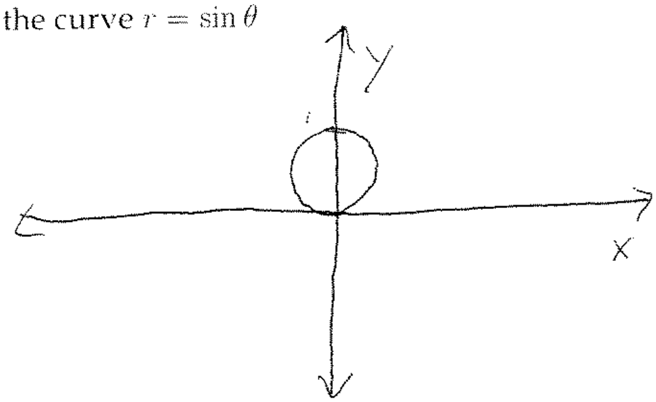
Polar to Cartesian

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Now, just as we can describe curves in $\mathbb{R}^2$ as function of the form $y = f(x)$, we can describe curves using polar coordinates as well, using functions of the form $r = f(\theta)$.

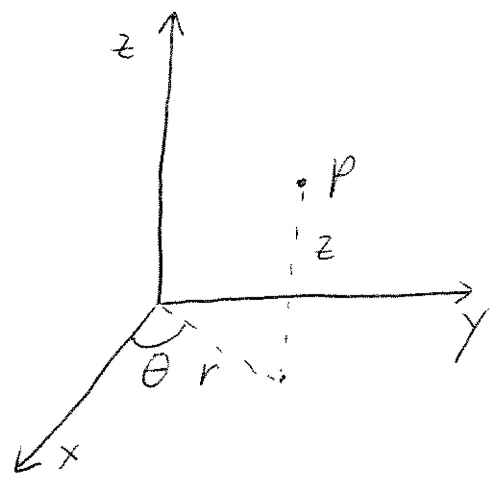
Example
Graph the curve $r = \sin \theta$



This idea of using different coordinate systems generalizes to three-space, and in fact can and does generalize to spaces of any dimension.

1.2 Cylindrical Coordinates

Cylindrical coordinates are a very natural generalization of polar coordinates. To describe a point (x, y, z) in three-space, we translate its x and y coordinates into the corresponding coordinates in polar coordinates, and then leave the z coordinate alone. Formally, we take a line segment connecting the point with the origin, project this segment onto the xy-plane, and then give the length of this projected segment, and the angle the segment makes with the positive x-axis. We then give the z value, which is the signed length of the line segment's projection onto the z-axis. You should read through that description and make sure it makes sense to you, but it's much easier to understand with a picture:



Now, just as in polar coordinates, we may wish to convert from Cartesian to cylindrical, or from cylindrical to Cartesian. The formulas for doing so follow from the corresponding formulas for polar coordinates:

Cartesian to Cylindrical:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

$$z = z$$

Cylindrical to Cartesian:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Example
Convert the point $(3, 4, 5)$ into cylindrical coordinates.

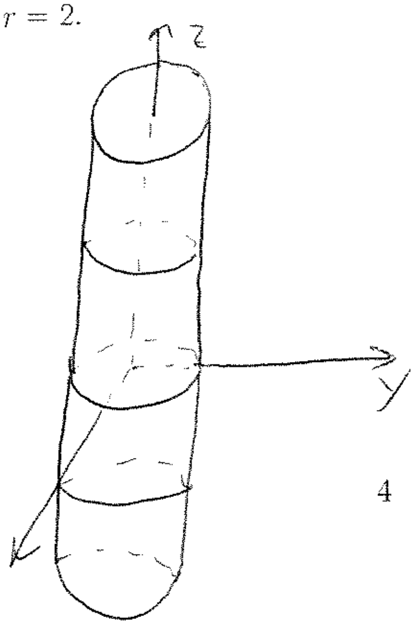
$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1^\circ$$

$$z = 5$$

Now, in analogy with two-dimensional Cartesian and polar coordinates, we can describe surfaces in $\mathbb{R}^3$ using Cartesian equations of the form $z = f(x, y)$ or using cylindrical equations of the form $r = f(\theta, z)$.

Example
Graph and describe the surface given by the cylindrical coordinate equation $r = 2$.

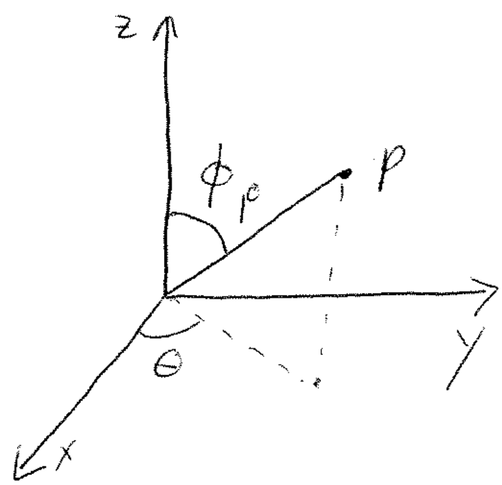


A right-circular cylinder of radius 2.

We note that for cylindrical coordinate the domain of r is $[0, \infty)$, the domain of θ is $[0, 2\pi)$, and the domain of z is the same as in Cartesian coordinates, namely, $(-\infty, \infty)$. We also note that if $r = 0$ we have some redundancy, in that cylindrical coordinates will give us the same point on the z-axis regardless of the value of θ, and so if $r = 0$ the value of θ becomes redundant.

1.3 Spherical Coordinates

In addition to Cartesian and cylindrical, there is one more common coordinate system used to identify points in three-dimensional space. This coordinate system is called spherical coordinates. The idea here is to describe a point by drawing a line segment from the origin to that point, and then specifying the length of this line segment. This gives us one value, which we denoate as ρ. The other two defining values for the point are the angle the line segment makes with the z-axis, denoted ϕ, and the angle the projection of the line segment onto the xy-plane makes with the positive x-axis.



Now, again, we have formulas for converting from Cartesian to spherical or back, and from cylindrical to spherical or back:

Cartesian to Spherical

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan \frac{y}{x}$$

$$\phi = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Spherical to Cartesian

$$x = \rho \cos \theta \sin \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \phi$$

Cylindrical to Spherical

$$\rho = \sqrt{r^2 + z^2}$$

$$\theta = \theta$$

$$\phi = \arccos \frac{z}{\sqrt{r^2 + z^2}}$$

Spherical to Cylindrical

$$r = \rho \sin \phi$$

$$\theta = \theta$$

$$z = \rho \cos \phi$$

Example
Convert the point $(3, 4, 5)$ into spherical coordinates.

$$\rho = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1^\circ$$

$$\phi = \cos^{-1}\left(\frac{5}{5\sqrt{2}}\right) = 45^\circ$$

We can also of course describe surfaces in spherical coordinates using equations of the form $\rho = f(\theta, \phi)$.

Example
Describe the graph of $\rho = 2\cos\phi$

$$\rho = 2\cos\phi$$

$$\Rightarrow \sqrt{x^2+y^2+z^2} = \frac{2z}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow x^2+y^2+z^2 = 2z$$

$$\Rightarrow x^2+y^2+(z-1)^2 = 1$$

So, a sphere of radius 1 centered at $(0,0,1)$

Example
Make the following conversions:

1. $x^2 - y^2 = 25$ to cylindrical coordinates.

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 25$$

$$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 25 \Rightarrow \boxed{2r^2\cos(2\theta) = 25}$$

2. $x^2+y^2-z^2=1$ to spherical coordinates.

$$(\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2-(\rho\cos\phi)^2=1$$

$$\Rightarrow \rho^2(\cos^2\theta+\sin^2\theta)\sin^2\phi-\rho^2\cos^2\phi=1$$

$$\Rightarrow \rho^2(\sin^2\phi-\cos^2\phi)=1 \Rightarrow \boxed{-2\rho^2\cos(2\phi)=1}$$

3. $\rho=2\cos\phi$ to cylindrical coordinates.

$$\sqrt{r^2+z^2}=\frac{2z}{\sqrt{r^2+z^2}}$$

$$\Rightarrow r^2+z^2=2z$$

$$\Rightarrow \boxed{r^2+(z-1)^2=1}$$

4. $x+y+z=1$ to spherical coordinates.

$$\rho\cos\theta\sin\phi+\rho\sin\theta\sin\phi+\rho\cos\phi=1$$

$$\Rightarrow \boxed{\rho=\frac{1}{\cos\theta\sin\phi+\sin\theta\sin\phi+\cos\phi}}$$

5. $r=2\sin\theta$ to Cartesian coordinates.

$$\sqrt{x^2+y^2}=\frac{2y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow x^2+y^2=2y$$

$$\Rightarrow \boxed{x^2+(y-1)^2=1}$$

We note finally that the domain of ρ is $[0,\infty)$, the domain of θ is $[0,2\pi)$, and the domain of ϕ is $[0,\pi]$. We note also that if $\rho=0$ then θ and φ do not need to be specified, and if $\varphi = 0$ or $\varphi = \pi$ then θ does not need to be specified. I keep harping on this fact because these kind of considerations actually become important, and interesting, in later mathematics classes that deal with things called manifolds.