Math 2210 - Section 11.8 Notes

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1 Surfaces in Three-Space

1.1 Basic Concepts

The graph of a 3-variable equation in three dimensions is usually a surface. We've seen examples of this already, namely, the plane: $Ax + By + Cz = D$ and the sphere: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

Graphing surfaces on a piece of paper (or on a blackboard) is very hard, believe me, but there are ways of visualizing these surfaces that make it easier. Frequently, we can construct an illustrative picture by finding the intersections of the surface with well-chosen planes. These inersections are called *cross sections*. A special type of cross section is the intersection with one of the coordinate planes (the xy, xz, or yz planes). In these cases the cross section is called a *trace*.

Example 1
Sketch a graph of the surface in three-space defined by the equation:

$$y^2 + z^2 = 15.$$

1.2 Cylinders

This is one of the more difficult things for students in calculus III to understand, because we're about to redefine (or, more precisely, enlarge) the term *cylinder*. What we think of as a cylinder from high school geometry is actually a special type of cylinder called a right circular cylinder. The formal definition of a cylinder is:

Definition
Let C be a plane curve in three-space, and let l be a line intersecting C that is not in the plane of C. The set of all points on lines that are parallel to l and that intersect C is called a *cylinder*.

This is a much more general definition. A right circular cylinder is the cylinder whose plane curve is a circle, and whose line l is perpendicular to the plane of the associated circle. The surface we graphed in example 1 is an example of a right circular cylinder. Here is an example of a cylinder that isn't a right circular cylinder:

Example 2
Graph the surface defined by the equation:

$$x - z^2 = 0$$

1.3 Quadric Surfaces

A quadric surface is a surface in three-space whose defining equation is a second degree polynomial. The general form of a second degree polynomial of three variables is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Now, before you get afraid, don't panic. You won't be expected to deal with quadric surfaces in this type of generality. (Seriously, a variable J?)

The nice thing about quadric surfaces, that can be proven but that we'll just take on trust, is that through a rotation and translation, any quadric surface (we note that for a quadric surface we require $A^2 + B^2 + C^2 > 0$) can be transformed into one of the following two types of equation:

1. $Ax^2 + By^2 + Cz^2 + D = 0$

   or

2. $Ax^2 + By^2 + Cz = 0$

Now, the type of quadric surface is determined by the relative signs of the coefficients. A list of these possibilities, along with the corresponding equations and example graphs, is given at the end of these notes, and can be found in your textbook.

Example 3
Analyze the equation:

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

and sketch its graph.

Example 4

Name these graphs:

1. $9x^2 + y^2 - 16z^2 = -25$
2. $9x^2 + y^2 - 16z^2 = 25$
3. $x^2 + 4y^2 - 100z = 0$
4. $x^2 - y^2 = 0$
5. $x^2 - y^2 = 25$

Quadric Surfaces If a surface is the graph in three-space of an equation of second degree, it is called a **quadric surface.** Plane sections of a quadric surface are conics.

The general second-degree equation has the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

It can be shown that any such equation can be reduced, by rotation and translation of coordinate axes, to one of the two forms

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

The quadric surfaces represented by the first of these equations are symmetric with respect to the coordinate planes and the origin. They are called **central quadrics.**

In Figures 7 through 12, we show six general types of quadric surfaces. Study them carefully. The graphs were drawn by a technical artist; we do not expect that most of our readers will be able to duplicate them in doing the problems. A more reasonable drawing for most people to make is like the one that is shown in Figure 13 with our next example.

QUADRIC SURFACES

ELLIPSOID: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Plane	Cross Section
xy-plane	Ellipse
xz-plane	Ellipse
yz-plane	Ellipse
Parallel to xy-plane	Ellipse, point, or empty set
Parallel to xz-plane	Ellipse, point, or empty set
Parallel to yz-plane	Ellipse, point, or empty set

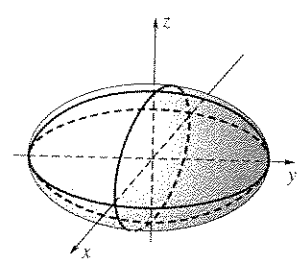

Figure 7

HYPERBOLOID OF ONE SHEET: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Plane	Cross Section
xy-plane	Ellipse
xz-plane	Hyperbola
yz-plane	Hyperbola
Parallel to xy-plane	Ellipse
Parallel to xz-plane	Hyperbola
Parallel to yz-plane	Hyperbola

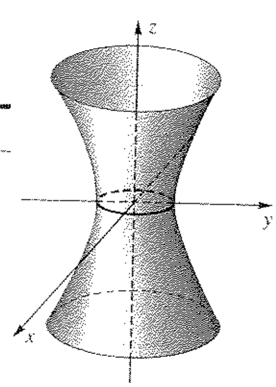

Figure 8

QUADRIC SURFACES (continued)

HYPERBOLOID OF TWO SHEETS: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Plane	Cross Section
xy-plane	Hyperbola
xz-plane	Hyperbola
yz-plane	Empty set
Parallel to xy-plane	Hyperbola
Parallel to xz-plane	Hyperbola
Parallel to yz-plane	Ellipse, point, or empty set

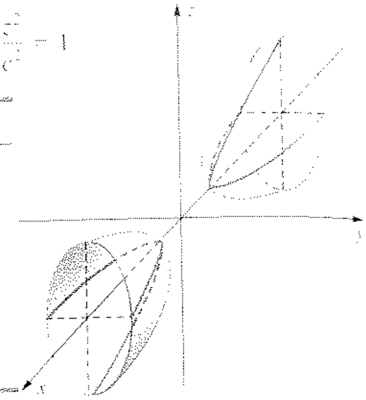

Figure 9

ELLIPTIC PARABOLOID: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Plane	Cross Section
xy-plane	Point
xz-plane	Parabola
yz-plane	Parabola
Parallel to xy-plane	Ellipse, point, or empty set
Parallel to xz-plane	Parabola
Parallel to yz-plane	Parabola

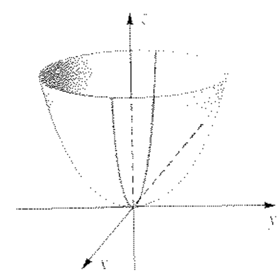

Figure 10

HYPERBOLIC PARABOLOID: $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$

Plane	Cross Section
xy-plane	Intersecting straight lines
xz-plane	Parabola
yz-plane	Parabola
Parallel to xy-plane	Hyperbola or intersecting straight lines
Parallel to xz-plane	Parabola
Parallel to yz-plane	Parabola

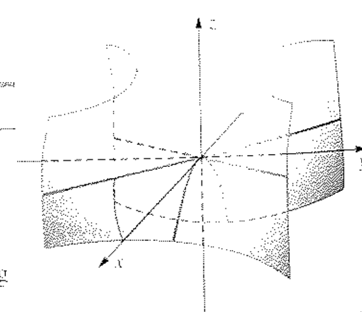

Figure 11

ELLIPTIC CONE: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

Plane	Cross Section
xy-plane	Point
xz-plane	Intersecting straight lines
yz-plane	Intersecting straight lines
Parallel to xy-plane	Ellipse or point
Parallel to xz-plane	Hyperbola or intersecting straight lines
Parallel to yz-plane	Hyperbola or intersecting straight lines

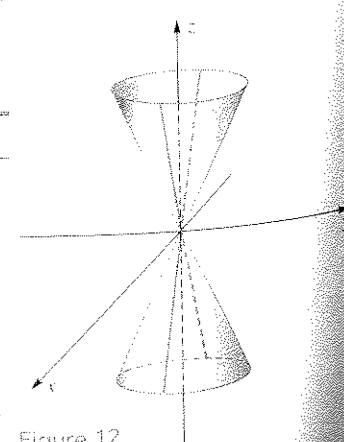

Figure 12