Math 2210 - Section 11.6 Notes

Dylan Zwick

Fall 2008

1 Lines and Tangent Lines in Three-Space

1.1 Definitions

The line is the simplest of all curves. We can describe a line by giving a fixed point $P_0 = (x_0, y_0, z_0)$ and a fixed vector $\mathbf{v} = < a, b, c >$ called the *direction vector* for the line. The line is the set of all points $P = (x, y, z)$ such that the vector from P_0 to P is a scalar multiple of $\mathbf{v}$.

We can express the line as a vector valued function of the form:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}_0$ and $\mathbf{v}$ are constant vectors. We can also represent the components of this vector valued function as linear equations:

$$\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned}$$

where $\mathbf{r}_0 = < x_0, y_0, z_0 >$ and $\mathbf{v} = < a, b, c >$. These are called the *parametric equations* of the line through the point (x_0, y_0, z_0) with *direction numbers* $< a, b, c >$. We note that these direction numbers are not unique. Any constant multiple $k < a, b, c >$ would also work, as long as $k \neq 0$.

Example 1 Find parametric equations for the line through the points $(2, -1, -5)$ and $(7, -2, 3)$.

$$(x_0, y_0, z_0) = (2, -1, -5)$$

$$\vec{v} = \langle a, b, c \rangle = \langle 7-2, -2-(-1), 3-(-5) \rangle = \langle 5, -1, 8 \rangle$$

$$x(t) = 2 + 5t$$
$$y(t) = -1 - t$$
$$z(t) = -5 + 8t$$

1.2 Symmetric Equations

We note that the parameter t appears in each of the three parametric equations. We can solve for t in each of these equations to get the trio of relations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This trio of equations also defined a line, where the line is given by the set of all points (x, y, z) that satisfy the above relations. Note that this assumes that none of a, b, c are zero.

Example 2 Write the symmetric equations for the line through $(-2, 2, -2)$ and parallel to $< 7, -6, 3 >$.

$$\frac{x - (-2)}{7} = \frac{y - 2}{-6} = \frac{z - (-2)}{3}$$

$$\Rightarrow \boxed{\frac{x+2}{7} = -\frac{y-2}{6} = \frac{z+2}{3}}$$

Example 3 Find the symmetric equations for the line of intersection between the planes:

$$x + y - z = 2$$
and
$$3x - 2y + z = 3$$

$\vec{n}_1 = \langle 1, 1, -1 \rangle$

$\vec{n}_2 = \langle 3, -2, 1 \rangle$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 3 & -2 & 1 \end{vmatrix} = (1(1) - (-1)(-2))\hat{i} + (-1(3) - 1(1))\hat{j} + (1(-2) - 1(3))\hat{k}$$

$$= \langle -1, -4, -5 \rangle$$

$z = 1$

$x + y = 3$

$3x - 2y = 2 \Rightarrow x = \frac{8}{5}, \quad y = \frac{7}{5}$

$5x = 8$

So, $(\frac{8}{5}, \frac{7}{5}, 1)$ is a point on the line:

$$\boxed{\begin{aligned} x(t) &= \tfrac{8}{5} - t \\ y(t) &= \tfrac{7}{5} - 4t \\ z(t) &= 1 - 5t \end{aligned}}$$

1.3 Tangent Lines to a Curve

If we let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be the vector-valued function of a curve in three-space, then the tantent line to the curve at any point $(f(t), g(t), h(t))$ has direction vector $< f'(t), g'(t), h'(t)$ and so the tangent line at t_0 has the equation:

$$x(t) = f(t_0) + tf'(t_0)$$
$$y(t) = g(t_0) + tg'(t_0)$$
$$z(t) = h(t_0) + th'(t_0)$$

Example 4

Find the parametric equations of the tangent line to the curve $x(t) = 2t^2$, $y(t) = 4t$, and $z(t) = t^3$, at time $t = 1$.

$$\begin{aligned} x'(t) &= 4t \\ y'(t) &= 4 \\ z'(t) &= 3t^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x'(1) &= 4 \\ y'(1) &= 4 \\ z'(1) &= 3 \end{aligned}$$

$$\begin{aligned} x(1) &= 2 \\ y(1) &= 4 \\ z(1) &= 1 \end{aligned}$$

So, the parametric equations are:

$$\boxed{\begin{aligned} x(t) &= 2 + 4t \\ y(t) &= 4 + 4t \\ z(t) &= 3 + t \end{aligned}}$$