Math 2210 - Section 11.5 Notes

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1 Vector-Valued Functions and Curvilinear Motion

1.1 Basic Concepts

A *vector-valued function* is a function F of $t \in \mathbb{R}$ that associates with every input t an output vector $\mathbf{F}(t)$. In $\mathbb{R}^3$ such a function would take the form:

$$\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where $f(t), g(t), h(t)$ are real valued function of the variable t.

Just as with real valued functions we have a concept of a limit for vector-valued functions:

Definition

We say that $\lim_{t \to c} \mathbf{F}(t) = \mathbf{L}$ means that for all $\epsilon > 0$, there is a $\delta > 0$ such that $||\mathbf{F}(t) - \mathbf{L}|| < \epsilon$, provided that $0 < |t - c| < \delta$. In more mathematical terminology:

$$0 < |t - c| < \delta \rightarrow ||\mathbf{F}(t) - \mathbf{L}|| < \epsilon.$$

Now, the definition of $\mathbf{F}(t)$ in terms of real valued component functions means that concepts such as limits, continuity, differentiation, and integration all carry over pretty much directly from single variable calculus. You just apply the operation to each of the component functions. So, we get the following theorem, which we state but do not prove (the proof can be found in the textbook):

Theorem
Let $\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. Then $\mathbf{F}$ has limit at c if and only if $f(t), g(t)$ and $h(t)$ have limits at c. If the limit exists then:

$$\lim_{t\to c}\mathbf{F}(t) = [\lim_{t\to c} f(t)] + [\lim_{t\to c} g(t)] + [\lim_{t\to c} h(t)]$$

The definition of continuous and differentiable is what we'd expect from single variable calculus:

Definition
A vector-valued function is *continuous* at $c \in \mathbb{R}$ if:

$$\lim_{t\to c}\mathbf{F}(t) = \mathbf{F}(c).$$

A derivative of a vector-valued function at a point $c \in \mathbb{R}$ is defined as:

$$\mathbf{F}'(t) = \lim_{h\to 0}\frac{\mathbf{F}(t+h) - \mathbf{F}(t)}{h}.$$

If this limit is well defined at a point $c \in \mathbb{R}$ then we say that the vector-valued functiion $\mathbf{F}$ is *differentiable* at the point c.

We note that, in practice, differentiation just means termwise differentiation. So, for example, in $\mathbb{R}^3$ we have the relation:

$$\mathbf{F}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

From this termwise relation we can pretty easily prove the following differentiation formulas:

1. $D_t[\mathbf{F}(t) + \mathbf{G}(t) = \mathbf{F}'(t) + \mathbf{G}'(t)$
2. $D_t[c\mathbf{F}(t)] = c\mathbf{F}'(t)$
3. $D_t[p(t)\mathbf{F}(t)] = p(t)\mathbf{F}'(t) + p'(t)\mathbf{F}(t)$
4. $D_t[\mathbf{F}(t)\cdot\mathbf{G}(t)] = \mathbf{F}(t)\cdot\mathbf{G}'(t) + \mathbf{F}'(t)\cdot\mathbf{G}(t)$

5. $D_t[\mathbf{F}(t) \times \mathbf{G}(t)] = \mathbf{F}(t) \times \mathbf{G}'(t) + \mathbf{F}'(t) \times \mathbf{G}(t)$

6. $D_t[\mathbf{F}(p(t))] = \mathbf{F}'(p(t))p'(t)$

Finally, we note that as differentiation is termwise, we define integration termwise as well:

$$\int \mathbf{F}(t)dt = [\int f(t)dt]\mathbf{i} + [\int g(t)dt]\mathbf{j} + [\int h(t)dt]\mathbf{k}$$

Example 1

Calculate $\lim_{t\to\infty}[\frac{t\sin t}{t^2}]\mathbf{i} - [\frac{7t^3}{t^3 - 3t}]\mathbf{j}$

$$\lim_{t\to\infty} \frac{t\sin t}{t} = \lim_{t\to\infty} \frac{\sin t}{t}$$

$$\lim_{t\to\infty} -\frac{1}{t} \leq \lim_{t\to\infty} \frac{\sin t}{t} \leq \lim_{t\to\infty} \frac{1}{t} = 0$$

$$\lim_{t\to\infty} -\frac{7t^3}{t^3-3t} = -7, \quad \text{So, } \lim_{t\to\infty}\left(\frac{t\sin t}{t^2}\hat{i} - \frac{7t^3}{t^3-3t}\hat{j}\right) = \boxed{-7\hat{j}}$$

Example 2

Find $\mathbf{r}'(x)$ and $\mathbf{r}''(x)$ for:

$$\mathbf{r}(x) = (e^x + e^{-x^2})\mathbf{i} + 2^x\mathbf{j}$$

$$\vec{r}'(x) = \left(e^x - 2xe^{-x^2}\right)\hat{i} + 2^x \ln 2\,\hat{j}$$

$$\vec{r}''(x) = \left(e^x + 4x^2e^{-x^2} - 2e^{-x^2}\right)\hat{i} + 2^x(\ln 2)^2\hat{j}$$

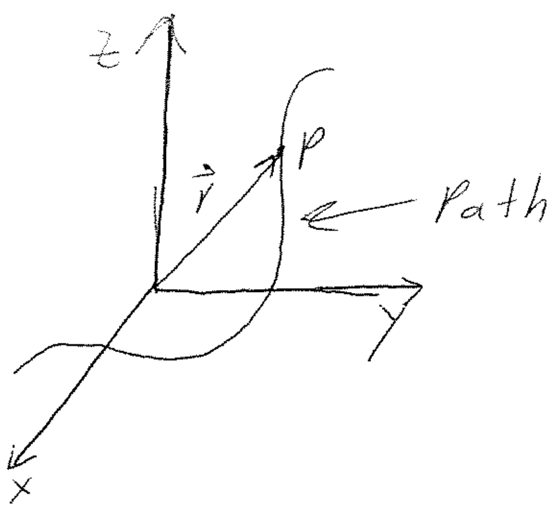
1.2 Curvilinear Motion

If $\mathbf{r}(t)$ is a *position vector* at any time t along a curve given by:

$$x = x(t),\ y = y(t), \text{ and } z = z(t)$$

then $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

Now, as the time t varies, the head of $\mathbf{r}(t)$ traces the path of the moving point P (pictured below). The path it traces out is a curve, and the corresponding motion is called *curvilinear motion*

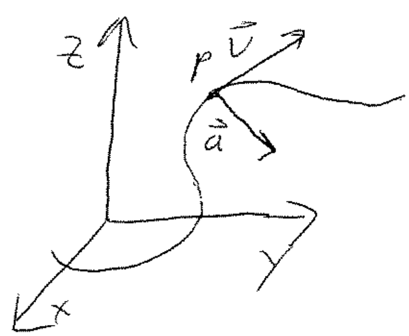


Now, the *velocity* and *acceleration* of this motion are defined as being the first and second derivatives, respectively, of the position vector $\mathbf{r}(t)$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$



Now, the velocity vector always points tangent to the curve, while the acceleration vector points towards the "concave" side of the curve.

Example 3
Given $\mathbf{r}(t) = 4\sin t\mathbf{i} + 8\cos t\mathbf{j}$, find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

$$\vec{v}(t) = 4\cos t\,\hat{i} - 8\sin t\,\hat{j}$$

$$\vec{a}(t) = -4\sin t\,\hat{i} - 8\cos t\,\hat{j}$$