Math 2210 - Section 11.4 Notes

Dylan Zwick

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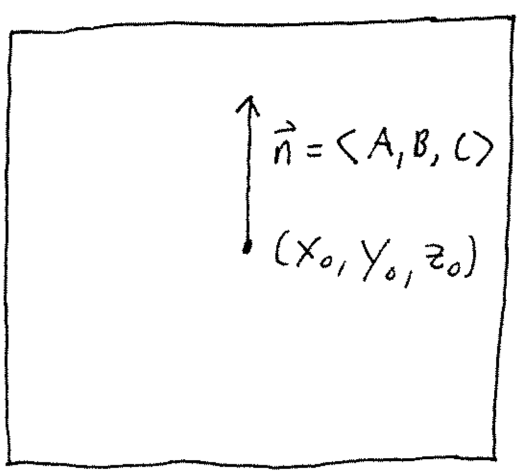
1 Planes, Points, and Cross Products

1.1 Planes

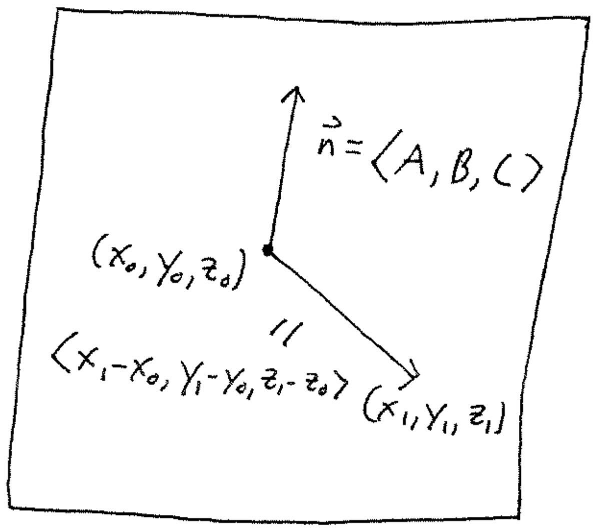
We said earlier that a plane in three dimensional space is defined by an equation of the form:

$$Ax + By + Cz = D$$
where
$$A^2 + B^2 + C^2 \neq 0$$

Another, equivalent, way of defining planes is to use vectors and dot products. For any given non-zero vector $\mathbf{n} = < A, B, C >$ and any fixed point (x_0, y_0, z_0) there will be a unique plane that contains the point and is perpendicular to the given vector.



This plane is the set of points that lie at the tip of vectors that begin at the point (x_0, y_0, z_0) and are perpendicular to the vector $\mathbf{n}$.



These vectors will have the form $< x - x_0, y - y_0, z - z_0 >$ and their defining characteristic will be that they are perpendicular to the vector $\mathbf{n}$, in other words $< x - x_0, y - y_0, z - z_0 > \cdot \mathbf{n} = 0$. So, the defining equation for the plane will be:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

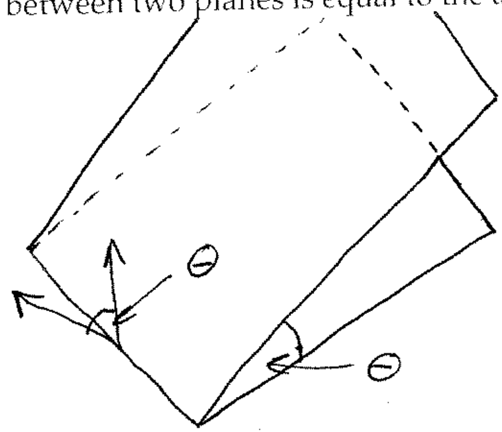
After some algebra we can see that this is equivalent to our standard equation:

$$Ax + By + Cz = D$$
with
$$D = Ax_0 + By_0 + Cz_0.$$

Example 1

Find the equation of the plane through the point $(1, -3, 4)$ that is perpendicular to the vector $\mathbf{n} =< -1, 2, -1 >$.

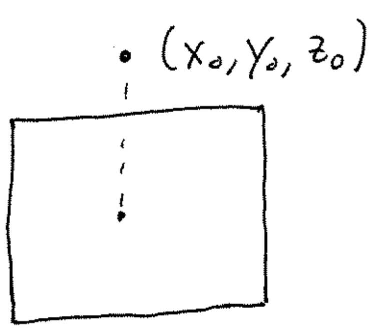
Now, the angle between two planes is equal to the angle between their normal vectors.



Example 2
Find the angle between the planes defined by the equations

$$2x + 4y + 3z = 8$$
and
$$3x - 4y + 7z = 5.$$

Now, the distance between a point and a plane is defined as the minimum distance between the point and all points in the plane.

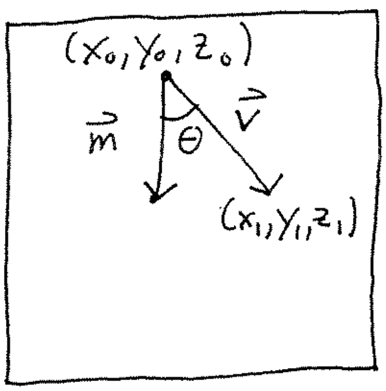


If we take any point (x_0, y_0, z_0) and any point (x_1, y_1, z_1) in the plane defined by $Ax + By + Cz = D$, then the distance between the point (x_0, y_0, z_0) and the plane will be the length of the projection of the vector

$$< x_1 - x_0, y_1 - y_0, z_1 - z_0 >$$

onto the normal vector

$$< A, B, C > .$$



$$\vec{m} = proj_{\vec{n}}(\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle)$$

$$\|\vec{m}\| = \|\vec{v}\| \cos\theta = \frac{|\vec{v} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}$$

Now, (x_1, y_1, z_1) satisfies $Ax_1 + By_1 + Cz_1 = D$, so

$$= \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$ ← Noting absolute value.

So, our formula for the distance from a point (x_0, y_0, z_0) to the plane $Ax + By + Cz = D$ is:

$$L = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example 3
Find the distance between parallel planes

$$-3x + 2y + z = 9$$

and

$$6x - 4y - 2z = 19.$$

1.2 The Cross Product

The *cross product* of two vectors **u** and **v** is a map that takes two vectors as input and outputs a single vector $\mathbf{u} \times \mathbf{v}$. The cross product is defined, in terms of components, as:

$$\mathbf{u} \times \mathbf{v} = < u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$$

This can be remembered as the determinant of the "matrix":

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example 4
Calculate $\mathbf{u} \times \mathbf{v}$ with $\mathbf{u} =< 1, -2, -2 >$ and $\mathbf{v} =< -2, 4, 1 >$.

Some properties of the cross product, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors, k is a scalar, and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$
3. $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin \theta$
4. $\mathbf{u} \times (\mathbf{u} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
5. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
6. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
7. $\mathbf{u} \times \mathbf{u} = 0$
8. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
9. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Example 5

Find the equation of the plane through the three points $(1, -2, 3)$, $(4, 1, -2)$, and $(-2, -3, 0)$.