Math 2210 - Section 11.3 Notes

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1 The Dot Product

1.1 Definitions

The *dot product* is a map from two vectors that produces a scalar. The dot product is also called the *scalar product*. In n dimensional space, $\mathbb{R}^n$, it is defined in terms of components as:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

So, in 2-dimensional space it is:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2,$$

while in 3-dimensional space it is:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The dot product has the following properties:

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{0} \cdot \mathbf{u} = 0$
5. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

1.2 The Dot Product and Angles

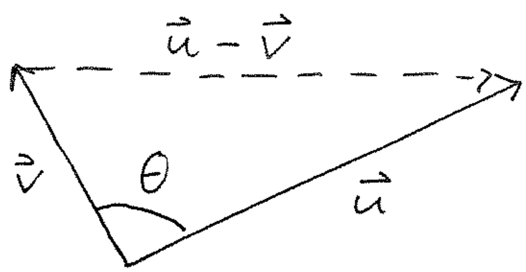
For two vectors $\mathbf{u}$ and $\mathbf{v}$ the dot product relates the angle between the two vectors:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Now, we note that if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (also called orthogonal) then $\theta = 90°$.

Proof



Apply the Law of Cosines:

$$|\mathbf{v} - \mathbf{u}|^2 = |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta.$$

On the other hand using the above properties we have:

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

Equating these two equations and performing some simple algebra we get:

$$\begin{aligned} |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v} \\ \to -2|\mathbf{u}||\mathbf{v}| \cos \theta &= -2\mathbf{u} \cdot \mathbf{v} \\ \to \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}||\mathbf{v}| \cos \theta. \end{aligned}$$

Example 1
For what numbers c are $< 2c, -8, 1 >$ and $< 3, c, c-2 >$ orthogonal?

$$\langle 2c, -8, 1\rangle \cdot \langle 3, c, c-2\rangle$$
$$= 6c - 8c + c - 2 = 0$$
$$\Rightarrow -c - 2 = 0 \Rightarrow \boxed{c = -2}$$

1.3 Direction Angles and Cosines

The smallest nonnegative angles between a nonzero three-dimensional vector **a** and the basis vectors **i**, **j**, and **k** are called the *direction angles* of **a**. They are denoted by α, β, and γ, respectively. If $\mathbf{a} = < a_1, a_2, a_3 >$ then:

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$
$$\cos\beta = \frac{\mathbf{a}\cdot\mathbf{j}}{|\mathbf{a}||\mathbf{j}|} = \frac{a_2}{|\mathbf{a}|}$$
$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{k}}{|\mathbf{a}||\mathbf{k}|} = \frac{a_3}{|\mathbf{a}|}$$

We note that:

$$(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1.$$

Example 2
Prove the above relation.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2}{|\vec{a}|^2} + \frac{a_2^2}{|\vec{a}|^2} + \frac{a_3^2}{|\vec{a}|^2}$$
$$= \frac{|\vec{a}|^2}{|\vec{a}|^2} = 1$$

Example 3
Find the direction cosines for $\mathbf{u} = < -1, 2, -2 >$.

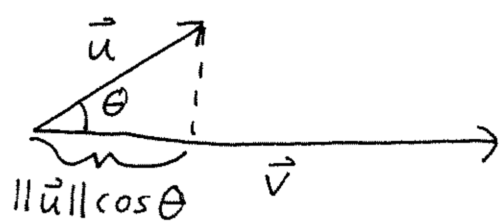
$$\|\vec{u}\| = \sqrt{(-1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$\cos\alpha = -\frac{1}{3}$$

$$\cos\beta = \frac{2}{3}$$

$$\cos\gamma = -\frac{2}{3}$$

1.4 Projections



Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Let $\mathbf{w}$ be the vector in the direction of $\mathbf{v}$ that has magnitude $|\mathbf{u}|\cos\theta$. Since $\mathbf{w}$ has the same direction as $\mathbf{v}$, we know that $\mathbf{w} = c\mathbf{v}$ for some nonnegative scalar c. This constant c is:

$$c = \frac{|\mathbf{u}|}{|\mathbf{v}|}\cos\theta = \frac{|\mathbf{u}|}{|\mathbf{v}|}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}.$$

Thus,

$$\mathbf{w} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

This vector $\mathbf{w}$ is called the *projection* of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$.

Example 4 Let $\mathbf{u} = < 1, 6, -2 >$ and $\mathbf{v} = < -3, 2, 5 >$. Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$pr_{\vec{v}}(\vec{u}) = \frac{\langle 1, 6, -2\rangle \cdot \langle -3, 2, 5\rangle}{\langle -3, 2, 5\rangle \cdot \langle -3, 2, 5\rangle} \langle -3, 2, 5\rangle$$

$$= \frac{-3+12-10}{9+4+25} \langle -3, 2, 5\rangle = \boxed{-\frac{1}{38} \langle -3, 2, 5\rangle}$$

We will postpone the discussion of planes until next time.