Math 2210 - Section 11.2 Notes

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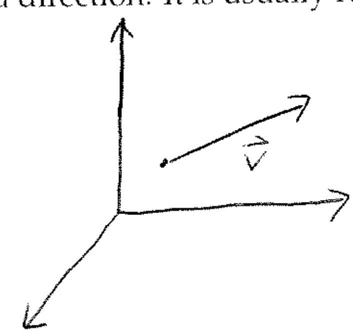
1 Vectors

1.1 The Basics

A scalar, as far as we're concerned, is just a real number ($\mathbb{R}$ again). It has a magnitude, but not a direction. A vector on the other hand is a quantity (magnitude) along with a direction. It is usually represented as an arrow:



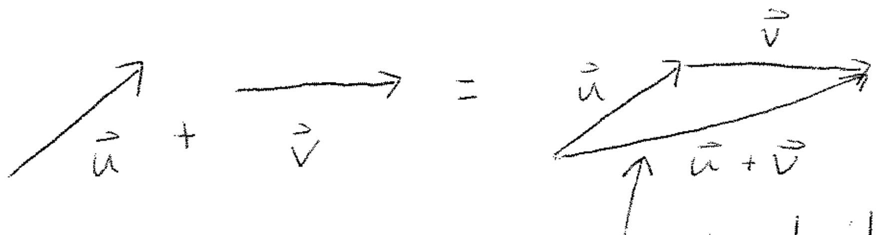
Note - What's important about a vector is its length and direction, **not** its starting point.

So, any two vectors that have the same magnitude and direction are considered the same, regardless of whether they have different starting points.

1.2 Adding Vectors Geometrically

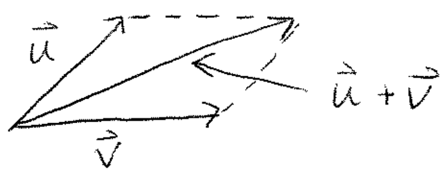
When dealing with vectors we have a concept of vector addition, and geometrically there are two standard ways of adding vectors (note that these two methods are essentially the same):

Tail - to - Tip $\mathbf{u}+\mathbf{v}$

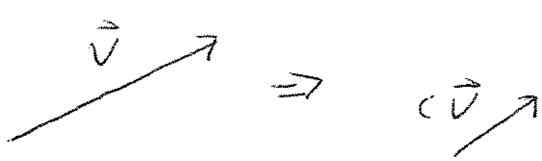


attach the tail of $\vec{v}$ to the tip of $\vec{u}$, then draw from the tail of $\vec{u}$ to the tip of $\vec{v}$.

Parallelogram $\mathbf{u}+\mathbf{v}$

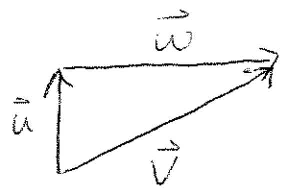


We can also multiply vectors by scalars. For a vector $\mathbf{v}$ if we multiply it by a scalar c we get a new vector $c\mathbf{v}$. This new vector has the same direction as $\mathbf{v}$, but its magnitude is the magnitude of $\mathbf{v}$ multiplied by c.



Example 1

Express the vector **w** in terms of **v** and **u**.

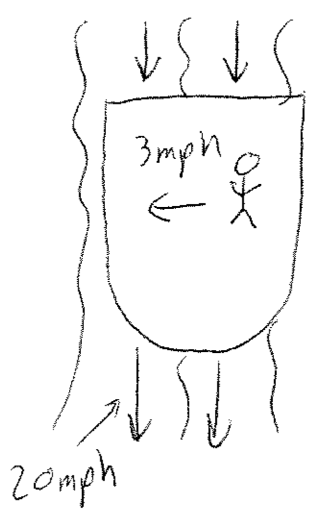


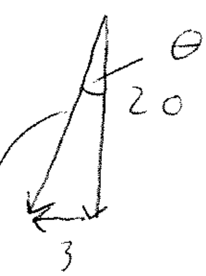
$$\Rightarrow \vec{v} = \vec{u} + \vec{w}$$

$$\Rightarrow \boxed{\vec{w} = \vec{v} - \vec{u}}$$

Example 2

A ship is sailing due south at 20 miles per hour. A man walks west across the deck at 3 miles per hour. What are the magnitude and direction of his velocity relative to the surface of the water?



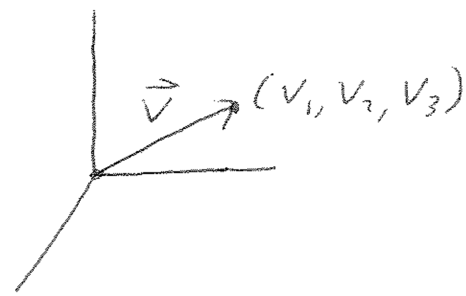


$$\theta = \tan^{-1}\left(\frac{3}{20}\right) = 8.5^\circ$$

magnitude 20.22

1.3 Algebraic Approach to Vectors

When we're dealing with a vector in either two-dimensions or three-dimensions we can represent a vector by its components $\mathbf{v} =< v_1, v_2, v_3 >$. For any vector, its components are the coordinates of its tip when the vector's tail is placed at the origin.



Now, vectors add component-wise.

Example 3
For the vectors represented by the components

$$\mathbf{v} =< 2, 4, 1 > \text{ and } \mathbf{u} =< 3, 1, 5 >$$

calculate the components of the vector $\mathbf{v} + \mathbf{u}$.

$$\vec{v} + \vec{u} = \langle 2+3, 4+1, 1+5 \rangle = \boxed{\langle 5, 5, 6 \rangle}$$

Now, the magnitude of a vector is its length, and its length is just given by the generalization of the Pythagorean theorem for distance in three dimensional space that we covered last lecture:

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

We can also deal with scalar multiplication of vectors (multiplication by scalars) component-wise:

$$c\mathbf{v} = c < v_1, v_2, v_3 > = < cv_1, cv_2, cv_3 >.$$

Now, a vector of length 1 is called a *unit vector*, and a vector with all 0 components is the *zero vector*, $\mathbf{0}$.

We also note that two vectors are the same if and only if they are the same in each of their components.

Theorem A

For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and all scalars $a, b \in \mathbb{R}$.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $a(b\mathbf{u}) = (ab)\mathbf{u} = \mathbf{u}(ab)$
6. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + b\mathbf{v}$
7. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Example 4

Let $\mathbf{u} = < -1, 5, 2 >$. Find $||\mathbf{u}||$, $|| -3\mathbf{u}||$. Also find a vector $\hat{\mathbf{u}}$, which is a vector in the same direction as $\mathbf{u}$, but with a magnitude of 1 (same direction, but a unit vector).

$$||\vec{u}|| = \sqrt{(-1)^2 + 5^2 + 2^2} = \boxed{\sqrt{30}}$$

$$||-3\vec{u}|| = \sqrt{3^2 + (-15)^2 + (-6)^2} = \sqrt{270} = \boxed{3\sqrt{30}}$$

$$\hat{u} = \frac{1}{\sqrt{30}} \langle -1, 5, 2 \rangle = \boxed{\left\langle -\frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}} \right\rangle}$$