Math 2210 - Section 11.1 Notes

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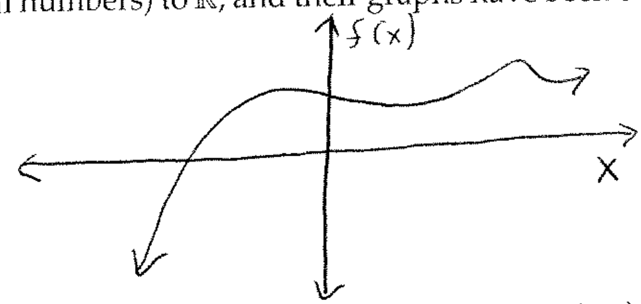
1 Cartesian Coordinates in Three Space

Thus far in calculus we've dealth pretty exclusively with 2-dimensional ideas and objects.

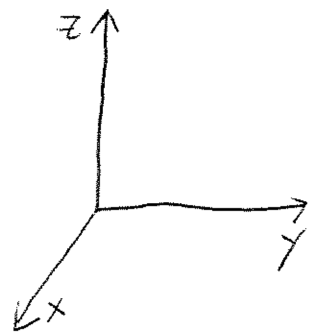
Our functions have been almost exclusively single variable maps from $\mathbb{R}$ (the real numbers) to $\mathbb{R}$, and their graphs have been on $\mathbb{R}^2$ (the xy-plane).



Points in $\mathbb{R}^2$ are represented by pairs of points (x,y).

Today, we cross the void into *THE THIRD DIMENSION*!!!
(Blows your mind, doesn't it...)

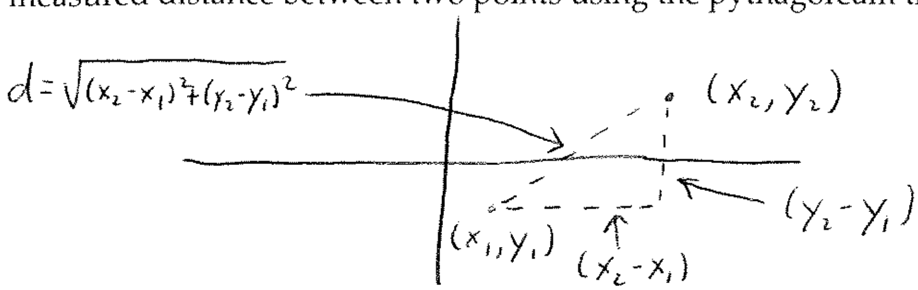
3-dimensional space (a.k.a. $\mathbb{R}^3$) is harder to draw than $\mathbb{R}^2$ (especially for me).



However, the basic ideas still apply. In $\mathbb{R}^3$ we specify a point with a triple of real coordinates, (x,y,z), which are the components in 3-space.

1.1 The Distance Formula

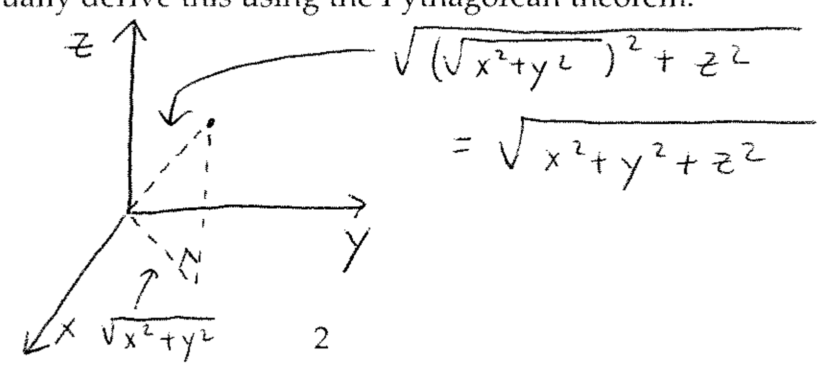
The first question we address is how to measure distance in $\mathbb{R}^3$. In $\mathbb{R}^2$ we measured distance between two points using the pythagoream theorem:



In $\mathbb{R}^3$ the distance formula extends in the natural way:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

We can actually derive this using the Pythagorean theorem:



Example 1 - Show that the points $(4,5,3)$, $(1,7,4)$, and $(2,4,6)$ are vertices of an equilateral triangle.

Distance from $(4,5,3)$ to $(1,7,4)$:

$$\sqrt{(4-1)^2+(5-7)^2+(3-4)^2} = \sqrt{9+4+1} = \sqrt{14}$$

Distance from $(4,5,3)$ to $(2,4,6)$:

$$\sqrt{(4-2)^2+(5-4)^2+(3-6)^2} = \sqrt{4+1+9} = \sqrt{14}$$

Distance from $(1,7,4)$ to $(2,4,6)$:

$$\sqrt{(1-2)^2+(7-4)^2+(4-6)^2} = \sqrt{1+9+4} = \sqrt{14}$$

All sides are of equal length, and so form an equilateral triangle.

1.2 Describing Shapes in $\mathbb{R}^3$

We can recall from analytic geometry that a circle is defined as the set of all points a distance r (the radius) from a point (x_1, y_1) (the center). The equation for this object is:

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

and for set values (x, y, r) the set of all pairs (x, y) satisfying the above equation is a circle.

Similarly in $\mathbb{R}^3$ we can describe a 3-dimensional sphere as the set of all points (x, y, z) that satisfy:

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

for radius r and center (x, y, z).

Note that if we expand the above equation we get:

$$x^2 + y^2 + z^2 + (-2x_1)x + (-2y_1)y + (-2z_1)z + x_1^2 + y_1^2 + z_1^2 - r^2 = 0.$$

Now, this equation has the general form (noting that x_1, y_1, z_1 and r are just numbers):

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

where G, H, I and J are numbers. In our example:

$$G = -2x_1,\ H = -2y_1,\ I = -2z_1, \text{ and } J = x_1^2 + y_1^2 + z_1^2 - r^2.$$

Example 2

Complete the square to find the center and radius of the sphere given by the equation:

$$x^2 + y^2 + z^2 + 2x - 6y - 10z + 34 = 0$$

$$x^2 + y^2 + z^2 + 2x - 6y - 10z + 34$$

$$= (x+1)^2 + (y-3)^2 + (z-5)^2 - 1$$

So, we have:

$$(x+1)^2 + (y-3)^2 + (z-5)^2 = \underline{1}$$

Center: $(-1, 3, 5)$

Radius: $\underline{1}$

1.3 Midpoint Rule

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are end points of a line segment, the mid-point $M(m_1, m_2, m_3)$ has coordinates:

$$m_1 = \frac{x_1 + x_2}{2}, m_2 = \frac{y_2 + y_1}{2}, m_3 = \frac{z_2 + z_1}{2}$$

1.4 Linear Equations

Also in $\mathbb{R}^2$ we have equations of the form

$$ax + by = c$$

which represents a line. *Note* - You may know the equation better as:

$$y = -\frac{a}{b}x + \frac{c}{d}$$

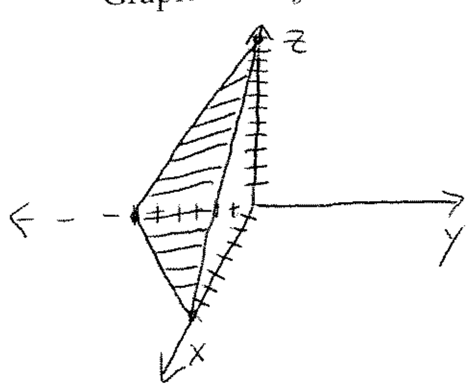
which is slope-intercept form (assuming $b \neq 0$).

The analog in $\mathbb{R}^3$ is not a line (although there are certainly lines in $\mathbb{R}^3$), but a plane. So, in $\mathbb{R}^3$ an equation of the form:

$$ax + by + cz = d \text{ where } a^2 + b^2 + c^2 \neq 0$$

represents a plane.

Example 3
Graph $3x - 4y + 2z = 24$.

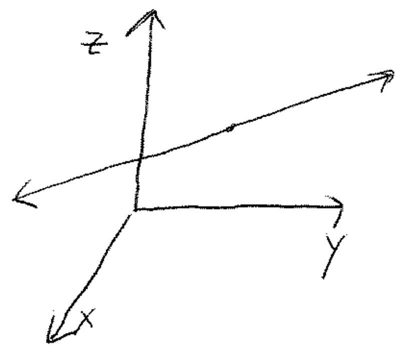


1.5 Parametric Representations of Curves in $\mathbb{R}^3$

I mentioned earlier that we can represent lines in $\mathbb{R}^3$. This is done with a parametric representation:

$$x(t) = at + b$$
$$y(t) = dt + d$$
$$z(t) = et + f$$

$$a, b, c, d, e, f \in \mathbb{R}$$



In fact, in general any curve in $\mathbb{R}^3$ can be (locally) represented as a trio of single variable functions:

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t).$$

The arc length formula transfers to $\mathbb{R}^3$ in the natural way:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}.$$

Example 4
Calculate the arc length from $1 \leq t \leq 2$ for the curve defined by:

$$x = t,\ y = \frac{2\sqrt{2}}{3}t^{\frac{3}{2}},\ z = \frac{1}{2}t^2$$

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = \sqrt{2}\, t^{1/2} \quad \frac{dz}{dt} = t$$

$$L = \int_1^2 \sqrt{1^2 + (\sqrt{2}\, t^{1/2})^2 + t^2}\, dt$$

$$= \int_1^2 \sqrt{1 + 2t + t^2}\, dt$$

$$= \int_1^2 \sqrt{(1+t)^2}\, dt$$

$$= \int_1^2 (1+t)\, dt$$

$$= t + \frac{t^2}{2} \Big|_1^2 = \left(2 + \frac{2^2}{2}\right) - \left(1 + \frac{1^2}{2}\right)$$

$$= 4 - \frac{3}{2} = \boxed{\frac{5}{2}}$$