Math 2210 - Exam 3

University of Utah

Fall 2008

Name: Solutions

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1. (20 points) *Multiple Integrals*

Calculate the following integrals:

(a) (3 points)

$$\int_0^{\pi}\int_0^1 x \sin y\,dx\,dy$$

$$\int_0^{\pi}\int_0^1 x \sin y\,dx\,dy = \int_0^{\pi} \frac{x^2}{2}\sin y\Big|_{x=0}^{x=1} dy$$

$$= \frac{1}{2}\int_0^{\pi} \sin y\,dy = -\frac{1}{2}\cos y\Big|_0^{\pi}$$

$$= -\frac{1}{2}(\cos\pi - \cos 0) = -\frac{1}{2}(-2)$$

$$= \boxed{1}$$

(b) (4 points)

$$\int_0^2\int_{-x}^{x} e^{-x^2}\,dy\,dx$$

$$\int_0^2\int_{-x}^{x} e^{-x^2}\,dy\,dx = \int_0^2 y e^{-x^2}\Big|_{y=-x}^{y=x} dx$$

$$= \int_0^2 2x e^{-x^2}\,dx$$

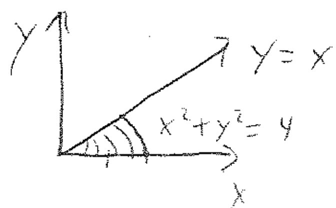
$$= -e^{-x^2}\Big|_0^2 = -e^{-4} - (-e^0) =$$

$$\boxed{1 - e^{-4}}$$

(c) (4 points)

$$\int\int_S \sqrt{4 - x^2 - y^2}\,dA$$

where S is the first quadrant sector of the circle $x^2 + y^2 = 4$ between $y = 0$ and $y = x$.



Converting to polar

$$\int_0^{\pi/4}\int_0^2 \sqrt{4 - r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} -\frac{(4-r^2)^{3/2}}{3}\Big|_{r=0}^{r=2} = \int_0^{\pi/4} \frac{8}{3}\,d\theta$$

$$= \frac{8}{3}\left(\frac{\pi}{4}\right) = \boxed{\frac{2\pi}{3}}$$

(d) (4 points)

$$\int_0^2\int_1^z\int_0^{\sqrt{\frac{x}{z}}} 2xyz\,dy\,dx\,dz$$

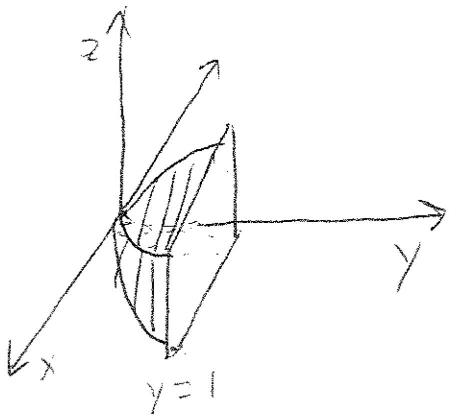
$$\int_0^2\int_1^z\int_0^{\sqrt{\frac{x}{z}}} 2xyz\,dy\,dx\,dz = \int_0^2\int_1^z xy^2z\Big|_0^{\sqrt{\frac{x}{z}}}\,dx\,dz$$

$$= \int_0^2\int_1^z x^2\,dx\,dz = \int_0^2 \frac{x^3}{3}\Big|_1^z\,dz$$

$$= \int_0^2\left(\frac{z^3}{3} - \frac{1}{3}\right)dz = \frac{z^4}{12} - \frac{z}{3}\Big|_0^2$$

$$= \frac{16}{12} - \frac{2}{3} = \frac{4}{3} - \frac{2}{3} = \boxed{\frac{2}{3}}$$

(e) (5 points)
Calculate the volume of the solid bounded by the cylinders $x^2 = y$ and $z^2 = y$ and the plane $y = 1$.



$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-\sqrt{y}}^{\sqrt{y}} dz\,dx\,dy$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 2\sqrt{y}\,dx\,dy$$

$$= \int_0^1 4y\,dy = 2y^2\Big|_0^1$$

$$= \boxed{2}$$

2. (8 points)

Find the minimum distance between the origin and the surface:

$$x^2y - z^2 + 9 = 0.$$

$d^2(x,y,z) = x^2 + y^2 + z^2 \qquad g(x,y,z) = x^2y - z^2 + 9$

$\nabla d^2 = \langle 2x, 2y, 2z \rangle \qquad \nabla g = \langle 2xy, x^2, -2z \rangle$

$2x = 2\lambda xy \quad \Rightarrow \quad 2x = -2xy \qquad y = -1, \ x = \pm\sqrt{2}$

$2y = \lambda x^2 \quad \Rightarrow \quad 2y = -x^2 \qquad$ or

$2z = -2\lambda z \quad \Rightarrow \lambda = -1 \qquad y = 0 \quad x = 0$

For $y = -1$, $x = \pm\sqrt{2}$

$z^2 = 7 \Rightarrow z = \pm\sqrt{7}$

$d(\pm\sqrt{2}, -1, \pm\sqrt{7}) = \sqrt{(\sqrt{2})^2 + 1^2 + (\sqrt{7})^2} = \sqrt{10}$

For $y = 0$, $x = 0$

$z^2 = 9 \Rightarrow z = \pm 3$

$d(0, 0, \pm 3) = \sqrt{3^2} = \boxed{3}$

So, a minimum at $(0, 0, \pm 3)$ with a distance of $\boxed{3}$

Note: If we use the relation $z^2 = x^2y + 9$ in $x^2 + y^2 + z^2$ we get $d^2(x,y) = x^2 + y^2 + x^2y + 9$ where we find $\frac{\partial d^2}{\partial x} = 2x + 2xy$ $\frac{\partial d^2}{\partial y} = 2y + x^2$ which is simultaneously 0 at $(0,0)$ and $(\pm\sqrt{2}, -1)$. We find $(0,0)$ is a min, while $(\pm\sqrt{2}, -1)$ is an inflection point.

3. (7 points)

Evaluate:

$$\int_0^{\sqrt{3}} \int_0^1 \frac{8x}{(x^2+y^2+1)^2} dy dx.$$

Hint: Reverse the order of integration.

Recall $\int \frac{1}{1+x^2} = \arctan x + C$

Reversing the order:

$$\int_0^1 \int_0^{\sqrt{3}} \frac{8x}{(x^2+y^2+1)^2} dx dy \qquad \begin{aligned} u &= x^2+y^2+1 \\ du &= 2x\, dx \end{aligned}$$

$$= \int_0^1 \int_{y^2+1}^{y^2+4} \frac{4\, du}{u^2} dy$$

$$= -\int_0^1 \frac{4}{u} \Big|_{u=y^2+1}^{u=y^2+4} dy = 4\int_0^1 \left(\frac{1}{y^2+1} - \frac{1}{y^2+4}\right) dy$$

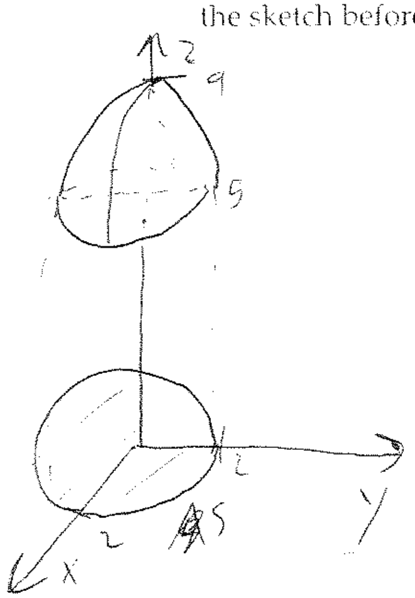
$$= 4\left(\tan^{-1}(y) - \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)\right)\Big|_0^1$$

$$= 4\tan^{-1}(1) - 2\tan^{-1}\left(\frac{1}{2}\right) = 4\left(\frac{\pi}{4}\right) - 2\tan^{-1}\left(\frac{1}{2}\right)$$

$$= \boxed{\pi - 2\tan^{-1}\left(\frac{1}{2}\right)}$$

4. (8 points)

Find the area of the surface that is the part of $z = 9 - x^2 - y^2$ above the plane $z = 5$. Make a sketch of the surface. (Probably best to make the sketch before calculating the surface area.)



$$\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial y} = -2y$$

$$SA = \iint_S \sqrt{1+4x^2+4y^2}\, dA$$

Converting to polar

$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\, r\,dr\,d\theta \qquad \begin{array}{l} u = 1+4r^2 \\ du = 8r\,dr \end{array}$$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^{17} \sqrt{u}\, du = \frac{1}{12}\int_0^{2\pi} u^{3/2}\Big|_1^{17}\, d\theta$$

$$= \frac{1}{12}(17\sqrt{17}-1)\int_0^{2\pi} d\theta$$

$$= \boxed{\frac{\pi(17\sqrt{17}-1)}{6}}$$

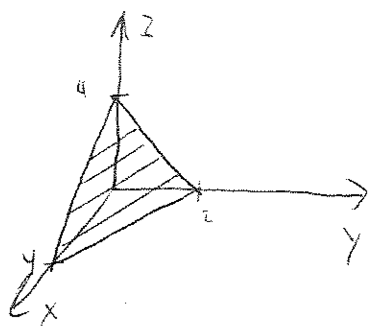
5. (7 points)

Write the iterated integral:

$$\int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} f(x, y, z)\,dx\,dz\,dy$$

as an integral with the order of integration $dz\,dy\,dx$.

Drawing the region



$x = 4 - 2y - z$

$z = 4 - 2y - x$

x goes from 0 to 4

y as a function of x goes from 0 to $2 - \frac{x}{2}$

z as a function of x and y goes from 0 to $4 - 2y - x$

$$\Rightarrow \boxed{\int_0^4 \int_0^{2-\frac{x}{2}} \int_0^{4-2y-x} f(x, y, z)\,dz\,dy\,dx}$$