Math 2210 - Exam 2

University of Utah

Fall 2008

Name: Solutions

1. (10 points) *Partial Derivatives*

   Calculate the partial derivatives, $f_x(x,y)$ and $f_y(x,y)$, of the following functions:

   (a) (3 points)

   $$f(x,y) = e^x \cos(y)$$

   $f_x(x,y) = e^x \cos(y)$

   $f_y(x,y) = -e^x \sin(y)$

(b) (3 points)

$$f(x, y) = y\cos(x^2 + y^2)$$

$f_x(x,y) = -2xy\sin(x^2+y^2)$

$f_y(x,y) = -2y^2\sin(x^2+y^2) + \cos(x^2+y^2)$

(c) (4 points)

$$f(x, y) = e^{x^2-y^2}$$

$f_x(x,y) = 2xe^{x^2-y^2}$

$f_y(x,y) = -2ye^{x^2-y^2}$

2. (10 points) *Limits*

Determine each of the following limits, or state it does not exist and give an explanation as to why:

(a) (3 points)

$$\lim_{(x,y)\to(-2,1)} (xy^3 - xy + 3y^2)$$

It's a polynomial, so we just plug in $(-2, 1)$:

$$-2(1)^3 - (-2)(1) + 3(1)^2 = \boxed{3}$$

(b) (3 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

Switch to polar:

$$\lim_{r\to 0} r\cos\theta\sin\theta = \boxed{0}$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{xy}{\sqrt{x^2+y^2}} = \frac{r^2\cos\theta\sin\theta}{r} = r\cos\theta\sin\theta$$

(c) (4 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

If we approach along $x=0$ we get:

$$\lim_{y\to 0} \frac{0}{y^4} = 0.$$

If we approach along $x=y^2$ we get:

$$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2} \neq 0$$

So, the limit does not exist, as we get different values for different paths of approach.

3. (15 points) *Gradients and Directional Derivatives*

For each problem, find the directional derivative of f at the point $\mathbf{p}$ in the direction of $\mathbf{a}$:

(a) (4 points)

$$f(x, y) = x^2 - 3xy + 2y^2$$
$$\mathbf{p} = (-1, 2),\ \mathbf{a} = 2\mathbf{i} - \mathbf{j}$$

$$\nabla f(x,y) = \langle 2x - 3y,\ -3x + 4y \rangle$$

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}$$

$$\nabla f(-1,2) = \langle -8,\ 11 \rangle$$

$$D_{\vec{a}} f(-1,2) = \langle -8, 11 \rangle \cdot \left\langle \frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}} \right\rangle = \boxed{-\frac{27}{\sqrt{5}}}$$

(b) (5 points)

$$f(x, y) = y^2 \ln x$$
$$\mathbf{p} = (1, 4),\ \mathbf{a} = \mathbf{i} - \mathbf{j}$$

$$\nabla f(x,y) = \left\langle \frac{y^2}{x},\ 2y \ln x \right\rangle$$

$$\nabla f(1,4) = \langle 16,\ 0 \rangle$$

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

$$\nabla f(1,4) \cdot \vec{u} = \langle 16, 0 \rangle \cdot \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle = \frac{16}{\sqrt{2}} = \boxed{8\sqrt{2}}$$

(c) (6 points)

Find the unit vector in the direction in which f increases most raplidly at $\mathbf{p}$. What is the rate of change in this direction?

$$f(x,y,z) = x^2yz$$
$$\mathbf{p} = (1,-1,2)$$

The direction of most rapid increase is the direction of the gradient $\nabla f(x,y,z)$.

$$\nabla f(x,y,z) = \langle 2xyz, x^2z, x^2y \rangle$$

$$\nabla f(1,-1,2) = \langle -4, 2, -1 \rangle$$

$$\|\nabla f(1,-1,2)\| = \sqrt{(-4)^2 + 2^2 + (-1)^2} = \sqrt{21}$$

$$\boxed{\vec{u} = \frac{1}{\sqrt{21}} \langle -4, 2, -1 \rangle}$$

The rate of change in this direction is:

$$\nabla f(1,-1,2) \cdot \vec{u} = \boxed{\sqrt{21}}$$

$$\|\nabla f(1,-1,2)\|$$

4. (5 points) *The Chain Rule*

The part of a tree normally sawed into lumber is the trunk, a solid shaped approximately like a right circular cylinder. If the radius of the trunk of a certain tree is growing $\frac{1}{2}$ inch per year and the height is increasing 8 inches per year, how fast is the volume increasing when the radius is 20 inches and the height is 400 inches? Express your answer in board feet per year (1 board foot = 1 inch by 12 inches by 12 inches).

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = 2\pi r h \left(\frac{dr}{dt}\right) + \pi r^2 \left(\frac{dh}{dt}\right)$$

$r = 20$ in $\qquad \frac{dr}{dt} = \frac{1}{2}$ in/year $\qquad \frac{dh}{dt} = 8$ in/year

$h = 400$ in

$$\frac{dV}{dt} = 2\pi(20)(400)\left(\frac{1}{2}\right) + \pi(20)^2(8)$$

$$= 8{,}000\pi \quad 3{,}200\pi = 11{,}200\pi \text{ in}^3/\text{year}$$

$$\frac{11{,}200\pi \text{ in}^3/\text{year}}{144 \text{ in}^3/\text{board foot}} = \boxed{\frac{700}{9}\pi \text{ board feet}/\text{year}}$$

5. (5 points) *Tangent Planes*

Find the equation of the tangent plane to the given surface at the indicated point.

$$x^2 - y^2 + z^2 + 1 = 0$$
$$\mathbf{p} = (1.3.\sqrt{7})$$

$$F(x, y, z) = x^2 - y^2 + z^2 + 1$$

$$\nabla F(x, y, z) = \langle 2x, -2y, 2z \rangle$$

$$\nabla F(1, 3, \sqrt{7}) = \langle 2, -6, 2\sqrt{7} \rangle$$

The tangent plane has equation:

$$2(x-1) - 6(y-3) + 2\sqrt{7}(z-\sqrt{7}) = 0$$

$$\Rightarrow \quad 2x - 2 - 6y + 18 + 2\sqrt{7}z - 14 = 0$$

$$\Rightarrow \quad 2x - 6y + 2\sqrt{7}z = -2$$

$$\boxed{x - 3y + \sqrt{7}z = -1}$$

6. (5 points) *Extrema*

Find all critical points of the given function, and indicate whether each such point gives a local maximum, a local minimum, or a saddle point.

$$f(x, y) = 2x^4 - x^2 + 3y^2$$

There is no boundary to the domain, all of $\mathbb{R}^2$, and as $f(x,y)$ is a polynomial it is differentiable everywhere.

$$\nabla f(x,y) = \langle 8x^3 - 2x,\ 6y \rangle$$

$$8x^3 - 2x = 0 \Rightarrow x = 0, \pm\tfrac{1}{2}$$

$$6y = 0 \Rightarrow y = 0$$

So, the stationary points are $(0,0)$, $(\tfrac{1}{2}, 0)$, and $(-\tfrac{1}{2}, 0)$.

$$D(x,y) = (24x^2 - 2)(6) - 0^2$$
$$= 144x^2 - 12$$

$D(0,0) = -12 \Rightarrow \boxed{f(0,0) = 0 \text{ is a saddle point.}}$

$D(\tfrac{1}{2}, 0) = 36 - 12 = 24 \Rightarrow \boxed{f(\tfrac{1}{2}, 0) = \tfrac{1}{8} - \tfrac{1}{4} = -\tfrac{1}{8} \text{ is a local min, as } f_{yy} = 6 > 0.}$

$D(-\tfrac{1}{2}, 0) = 36 - 12 = 24 \Rightarrow \boxed{f(-\tfrac{1}{2}, 0) = -\tfrac{1}{8} \text{ is a local min.}}$