Math 2210 - Exam 1

University of Utah

Fall 2008

Name: Solutions

1. For the vectors:

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

and

$$\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

calculate: (10 points)

(a) $\mathbf{a} + \mathbf{b}$ (3 points)

$$\vec{a} + \vec{b} = \langle 3+2, 4+1, -2+3 \rangle$$

$$= \langle 5, 5, 1 \rangle$$

or

$$5\hat{i} + 5\hat{j} + \hat{k}$$

(b) $\mathbf{a} \cdot \mathbf{b}$ (3 points)

$$\vec{a} \cdot \vec{b} = \langle 3, 4, -2 \rangle \cdot \langle 2, 1, 3 \rangle$$
$$= 6 + 4 - 6 = \boxed{4}$$

(c) $\mathbf{a} \times \mathbf{b}$ (4 points)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -2 \\ 2 & 1 & 3 \end{vmatrix}$$
$$= \boxed{14\hat{i} - 13\hat{j} - 5\hat{k}}$$

2. For the curve given by the vector equation:

$$\mathbf{r}(t) = t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$$

calculate: (20 points)

(a) The velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ (3 points)

$$\boxed{\vec{v}(t) = \hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}}$$

(b) The acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ (2 points)

$$\boxed{\vec{a}(t) = 0\,\hat{i} - \sin t\,\hat{j} - \cos t\,\hat{k}}$$

(c) The parametric equations of the tangent line to the curve at $t = \pi/4$. (4 points)

$$\vec{r}\left(\tfrac{\pi}{4}\right) = \frac{\pi}{4}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k} = \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}\left(\tfrac{\pi}{4}\right) = \hat{i} + \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}$$

So, $\left\langle 1, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$ is parallel to the line through the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$$\boxed{\begin{aligned} x(t) &= \frac{\pi}{4} + t \\ y(t) &= \frac{1}{\sqrt{2}} + \frac{t}{\sqrt{2}} \\ z(t) &= \frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}} \end{aligned}}$$

(d) The symmetric equations of this tangent line. (2 points)

$$\boxed{\frac{x - \frac{\pi}{4}}{1} = \frac{y - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{z - \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}}$$

(e) The plane containing the point $\mathbf{r}(\pi/4)$ that is perpendicular to the tangent line to the curve at that point. (4 points)

$$1\left(x-\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}\left(y-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(z-\frac{1}{\sqrt{2}}\right)=0$$

$$\Rightarrow \boxed{x+\frac{1}{\sqrt{2}}y-\frac{1}{\sqrt{2}}z=\frac{\pi}{4}}$$

(f) The length of the curve from $t=0$ to $t=\pi/4$. (4 points)

$$L=\int_0^{\pi/4}\sqrt{1^2+(\cos t)^2+(-\sin t)^2}\,dt$$

$$=\int_0^{\pi/4}\sqrt{2}\,dt$$

$$=\boxed{\frac{\sqrt{2}\,\pi}{4}}$$

3. Find symmetric equations of the line through $(4, 5, 8)$ and perpendicular to the plane $3x + 5y + 2z = 30$. (5 points)

A vector perpendicular to the plane is $\langle 3, 5, 2 \rangle$.

The symmetric equations are:

$$\boxed{\frac{x-4}{3} = \frac{y-5}{5} = \frac{z-8}{2}}$$

4. Make the following conversions: (10 points)

(a) Change the point given by $(6, \pi/6, -2)$ in cylindrical coordinates to its repersentation in Cartesian (rectangular) coordinates. (1 point)

$$x = 6\cos(\pi/6) \qquad y = 6\sin(\pi/6) \qquad z = -2 \quad \Rightarrow \quad \boxed{x = 3\sqrt{3},\ y = 3,\ z = -2}$$

(b) Change the point given by $(4, 4\pi/3, -8)$ in cylindrical coordinates to its representation in Cartesian (rectangular) coordinates. (1 point)

$$x = 4\cos\left(\frac{4\pi}{3}\right) \qquad y = 4\sin\left(\frac{4\pi}{3}\right) \qquad z = -8$$

$$\boxed{x = -2,\ y = -2\sqrt{3},\ z = -8}$$

(c) Change the point givey by $(8, \pi/4, \pi/6)$ in spherical coordinates to its representation in Cartesian (rectangular) coordinates. (1 point)

$$x = 8\cos(\pi/4)\sin(\pi/6) \qquad y = 8\sin(\pi/4)\sin(\pi/6) \qquad z = 8\cos(\pi/6)$$

$$\boxed{x = 2\sqrt{2},\ y = 2\sqrt{2},\ z = 4\sqrt{3}}$$

(d) Change the point givey by $(4, \pi/3, 3\pi/4)$ in spherical coordinates to its representation in Cartesian (rectangular) coordinates. (1 point)

$$x = 4\cos(\pi/3)\sin(3\pi/4) \qquad y = 4\sin(\pi/3)\sin(3\pi/4) \qquad z = 4\cos(3\pi/4)$$

$$\boxed{x = \sqrt{2},\ y = \sqrt{6},\ z = -2\sqrt{2}}$$

(e) Write the following Cartesian equation in cylindrical coordinate form: (2 points)

$$x^2 + y^2 = 9$$

$x^2+y^2=r^2$

$\Rightarrow \boxed{r^2=9}$

(f) Find the Cartesian equation corresponding to the following cylindrical coordinate equation: (2 points)

$$r^2 + z^2 = 9$$

$r^2 = x^2+y^2$

$\Rightarrow \boxed{x^2+y^2+z^2=9}$

(g) Write the following equation in spherical coordinates: (2 points)

$$x^2 + y^2 = z$$

$(\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 = \rho \cos\phi$

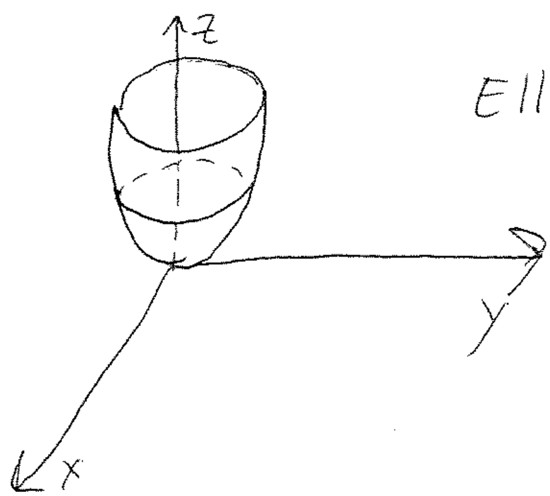
$\Rightarrow \rho^2 \sin^2\phi = \rho\cos\phi$

$\Rightarrow \rho = \dfrac{\cos\phi}{\sin^2\phi} = \left(\dfrac{\cos\phi}{\sin\phi}\right)\left(\dfrac{1}{\sin\phi}\right) = \cot\phi\csc\phi$

$\Rightarrow \boxed{\rho = \cot\phi\csc\phi}$

5. Graph and name the surface described by the quadric equation: (5 points)

$$z = x^2 + y^2$$



Elliptic Paraboloid