Math 2210 - Problem 13.8.21 Explanation

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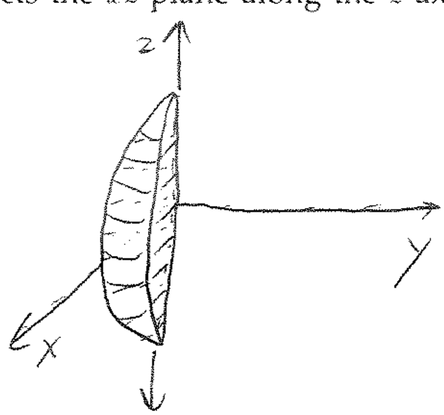
The problem is this:

Calculate the volume of the smaller wedge cut from the unit sphere by two planes that meet at a diameter at an angle of $30°$.

I was asked this today in class, and I screwed up on the answer. Sorry. Anyways, here's the right way to do it. Make one of the intersecting planes the xz-plane, and the other a plane that makes a $30°$ angle with the xz-plane and intersects the xz-plane along the z-axis to get something that looks like this:

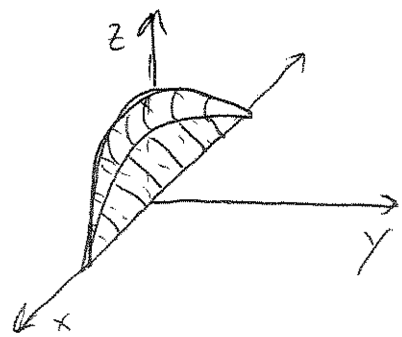


Then our limits of integration are easy, and we can calculate the volume:

$$V = \int_0^{\pi}\int_0^{\frac{\pi}{6}}\int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi = \frac{1}{3}\int_0^{\pi}\int_0^{\frac{\pi}{6}} \sin\phi d\theta d\phi$$
$$= \frac{\pi}{18}\int_0^{\pi} \sin\phi d\phi = \frac{\pi}{18}(-(-1)-(-1)) = \frac{\pi}{9}$$

So, that's how it's done.

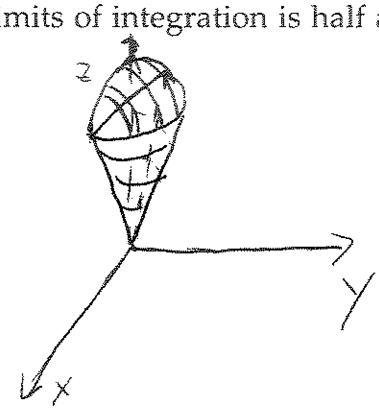
What I did was instead I said make one of the intersecting planes the xz-plane, and the other a plane that makes a $30°$ angle with the xz-plane and intersects the xz-plane along the x-axis to get something that looks like this:



Then I, incorrectly, said that the limits of integration would be:

$$\int_0^{\frac{\pi}{6}} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi$$

and I gave some BS explanations for why this is so. Again, sorry. The solid defined by these limits of integration is half a snowcone, and looks like this:



In fact, if you want to set it up this way and use spherical coordinates the limits of integration are incredibly nasty. If we integrate with respect to θ last, expressing ϕ as a function of θ we get:

$$\int_0^{\theta} \int_0^{\arcsin \sqrt{\frac{1}{1+3\sin\theta^2}}} \int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi$$

which works, although the integral becomes unspeakably nasty and more or less requires a computer to do.

If you insist upon having your outer variable be ϕ we note that ϕ goes from 0 to $\pi/2$, not 0 to $\pi/6$ as I said in class, and the integral must actually be broken up into two parts:

$$2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\arcsin \frac{\cot\phi}{\sqrt{3}}} \int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi + \int_0^{\frac{\pi}{6}} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi$$

which again will work, but will give you an unspeakably nasty integral. So, all things considered, this was definitely not the right approach. Again, sorry. If you want to know how I figured out the above limits please ask me after class or during my office hours.

-Dylan