Math 2210 - Assignment 9

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Sections 13.2 through 13.4

1 Section 13.2

13.2.1 Evaluate the integral:

$$\int_0^2 \int_0^3 (9-x)dydx$$

$$\int_0^2 \int_0^3 (9-x)\,dy\,dx = \int_0^2 \left[(9-x)y \Big|_{y=0}^{y=3} \right] dx$$

$$= \int_0^2 3(9-x)\,dx = 3\int_0^2 (9-x)\,dx$$

$$= 3\left[9x - \frac{x^2}{2}\right]\Big|_0^2 = 3(18-2) = \boxed{48}$$

13.2.8 Evaluate the integral:

$$\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y} dy dx$$

$$\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y} dy\, dx = \int_0^{\ln 3} \left[e^y \Big|_0^{\ln 2} \right] e^x dx$$

$$= \int_0^{\ln 3} (2-1) e^x dx = \int_0^{\ln 3} e^x dx$$

$$= e^x \Big|_0^{\ln 3} = 3-1 = \boxed{2}$$

13.2.14 Evaluate the integral:

$$\int_0^1 \int_0^2 \frac{y}{1+x^2}\,dy\,dx$$

$$\int_0^1 \int_0^2 \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \left(\frac{1}{1+x^2}\right)\left(\frac{y^2}{2}\right)\Bigg|_{0=y}^{2=y} dx$$

$$= 2 \int_0^1 \frac{1}{1+x^2}\,dx = 2\tan^{-1}(x)\Big|_0^1$$

$$= 2\left[\frac{\pi}{4} - 0\right] = \boxed{\frac{\pi}{2}}$$

13.2.20 Evaluate the indicated double integral over R.

$$\int\int_R xy\sqrt{1+x^2}dA$$

$$R = \{(x,y) : 0 \le x \le \sqrt{3}, 1 \le y \le 2\}$$

$$\int_0^{\sqrt{3}} \int_1^2 xy\sqrt{1+x^2}\, dy\, dx$$

$$= \int_0^{\sqrt{3}} x\sqrt{1+x^2}\left(\frac{y^2}{2}\right)\Bigg|_{y=1}^{y=2} dx$$

$$= \frac{3}{2}\int_0^{\sqrt{3}} x\sqrt{1+x^2}\, dx \qquad u = 1+x^2$$

$$\frac{du}{2} = x\,dx$$

$$= \frac{3}{4}\int_1^4 \sqrt{u}\, du$$

$$= \frac{3}{4}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{u=1}^{u=4}$$

$$= \frac{1}{2}(8-1) = \boxed{\frac{7}{2}}$$

13.2.41 Prove the **Cauchy-Schwarz Inequality for Integrals:**

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$$

Hint: Consider the double integral of:

$$F(x,y) = [f(x)g(y) - f(y)g(x)]^2$$

over the rectangle $R = \{(x,y) : a \leq x \leq b, a \leq y \leq b\}$.

We note $\quad 0 \leq [f(x)g(y) - f(y)g(x)]^2$

and so,

$$0 \leq \int_a^b\int_a^b [f(x)g(y) - f(y)g(x)]^2 dx\,dy$$

$$= \int_a^b\int_a^b f^2(x)g^2(y)\,dx\,dy - 2\int_a^b\int_a^b f(x)f(y)g(x)g(y)\,dx\,dy + \int_a^b\int_a^b f^2(y)g^2(x)\,dx\,dy$$

$$= \int_a^b f^2(x)dx \int_a^b g^2(y)dy - 2\int_a^b f(x)g(x)dx \int_a^b f(y)g(y)dy + \int_a^b f^2(y)dy \int_a^b g^2(x)dx$$

Note: $\int_a^b f^2(x)dx = \int_a^b f^2(y)dy$

$\int_a^b f(x)g(x)dx = \int_a^b f(y)g(y)dy$

$\int_a^b g^2(x)dx = \int_a^b g^2(y)dy$

$$\Rightarrow \cancel{0 \leq 2\int_a^b f^2(x)dx \int_a^b g^2(x)dx = 2\int_a^b\int_a^b}$$

$$\Rightarrow 0 \leq 2\int_a^b f^2(x)dx \int_a^b g^2(x)dx - 2\left[\int_a^b f(x)g(x)dx\right]^2$$

$$\Rightarrow \boxed{\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx}$$

2 Section 13.3

13.3.1 Evaluate the integral:

$$\int_0^1 \int_0^{3x} x^2 dy dx$$

$$\int_0^1 \int_0^{3x} x^2 dy dx = \int_0^1 \left[x^2 y \Big|_0^{3x} \right] dx$$

$$= \int_0^1 3x^3 dx = \frac{3}{4} x^4 \Big|_0^1$$

$$= \boxed{\frac{3}{4}}$$

13.3.6 Evaluate the integral:

$$\int_1^5 \int_0^x \frac{3}{x^2+y^2} dy dx$$

$$\int_1^5 \int_0^x \frac{3}{x^2+y^2} dy\, dx$$

$$= \int_1^5 \left[\frac{3}{x} \tan^{-1}\left(\frac{y}{x}\right) \Big|_{y=0}^{y=x} \right] dx$$

$$= \int_1^5 \frac{3}{x} \left(\frac{\pi}{4} - 0\right) dx$$

$$= \frac{3\pi}{4} \int_1^5 \frac{1}{x} dx = \frac{3\pi}{4}(\ln(5) - \ln(1))$$

$$= \boxed{\frac{3\pi}{4} \ln(5)}$$

13.3.11 Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin y} e^x \cos y dx dy$$

$$\int_0^{\pi/2} \int_0^{\sin y} e^x \cos y \, dx \, dy$$

$$= \int_0^{\pi/2} e^x \cos y \Big|_{x=0}^{x=\sin y} dy$$

$$= \int_0^{\pi/2} (e^{\sin y} - 1) \cos y \, dy \qquad u = \sin y$$

$$du = \cos y$$

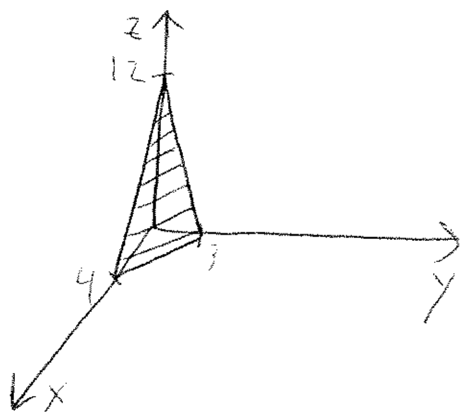
$$= \int_0^1 (e^u - 1) \, du$$

$$= e^u - u \Big|_0^1 = (e - 1) - (1 - 0)$$

$$= \boxed{e - 2}$$

13.3.22 Sketch the indicated solid, then find its volume by an iterated integration.

Tetrahedron bounded by the coordinate planes and the plane
$3x + 4y + z = 12.$



$$z = 12 - 3x - 4y$$

$$\int_0^4 \int_0^{3-\frac{3}{4}x} (12 - 3x - 4y)\, dy\, dx$$

$$= \int_0^4 (12y - 3xy - 2y^2) \Big|_{y=0}^{y=3-\frac{3}{4}x} dx$$

$$= \int_0^4 \left[12\left(3 - \frac{3}{4}x\right) - 3x\left(3 - \frac{3}{4}x\right) - 2\left(3 - \frac{3}{4}x\right)^2\right] dx$$

$$= \int_0^4 \left[36 - 9x - 9x + \frac{9}{4}x^2 - 18 + 9x - \frac{9}{8}x^2\right] dx$$

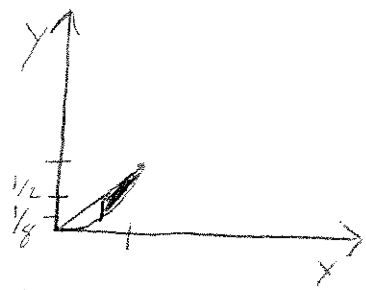
$$= \int_0^4 \left(18 - 9x + \frac{9}{8}x^2\right) dx = 18x - \frac{9}{2}x^2 + \frac{3}{8}x^3 \Big|_{x=0}^{x=4}$$

$$= 72 - 72 + 24 = \boxed{24}$$

13.3.36 Write the given iterated integral as an iterated integral with the order of integration interchanged.

$$\int_{\frac{1}{2}}^{1}\int_{x^3}^{x} f(x,y)\,dy\,dx$$

The region of integration is:



Switching this order we get:

$$\boxed{\int_{1/8}^{1/2}\int_{1/2}^{\sqrt[3]{y}} f(x,y)\,dx\,dy + \int_{1/2}^{1}\int_{y}^{\sqrt[3]{y}} f(x,y)\,dx\,dy}$$

3 Section 13.4

13.4.1 Evaluate the iterated integrals.

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} r^2 \sin\theta dr d\theta$$

$$\int_0^{\pi/2} \int_0^{\cos\theta} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3} \sin\theta \Big|_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\cos^3\theta \sin\theta}{3} d\theta \qquad \begin{aligned} u &= \cos\theta \\ du &= -\sin\theta \, d\theta \end{aligned}$$

$$= -\int_1^0 + \frac{u^3}{3} du = \int_0^1 \frac{u^3}{3} du$$

$$= \frac{u^4}{12} \Big|_0^1 = \boxed{\frac{1}{12}}$$

13.4.4 Evaluate the iterated integrals.

$$\int_0^{\pi}\int_0^{1-\cos\theta} r\sin\theta dr d\theta$$

$$\int_0^{\pi} \frac{r^2}{2}\sin\theta \Big|_{r=0}^{r=1-\cos\theta} d\theta$$

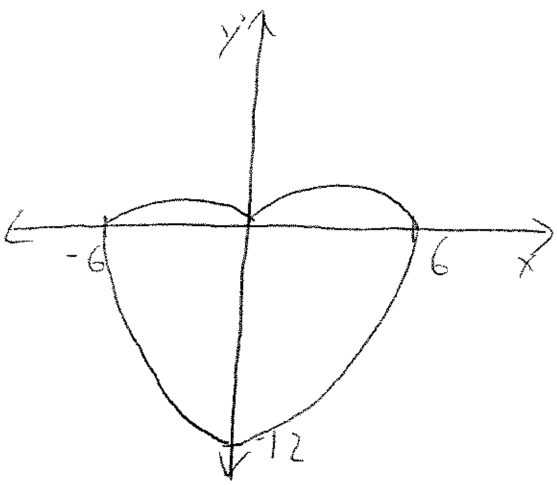
$$= \int_0^{\pi} \frac{(1-\cos\theta)^2}{2}\sin\theta d\theta \qquad \begin{aligned} u &= 1-\cos\theta \\ du &= \sin\theta d\theta \end{aligned}$$

$$= \int_0^{2} \frac{u^2}{2} du = \frac{u^3}{6}\Big|_0^2 = \boxed{\frac{4}{3}}$$

13.4.10 Find the area of the given region S by calculating $\int\int_S r\,dr\,d\theta$. Make a sketch of the region first.

S is the region inside the cardioid $r = 6 - 6\sin\theta$.

Sketch



The area is:

$$\int_0^{2\pi}\int_0^{6-6\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{6-6\sin\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(6-6\sin\theta)^2}{2}\,d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \int_0^{2\pi}(18 - 36\sin\theta + 18\sin^2\theta)\,d\theta$$

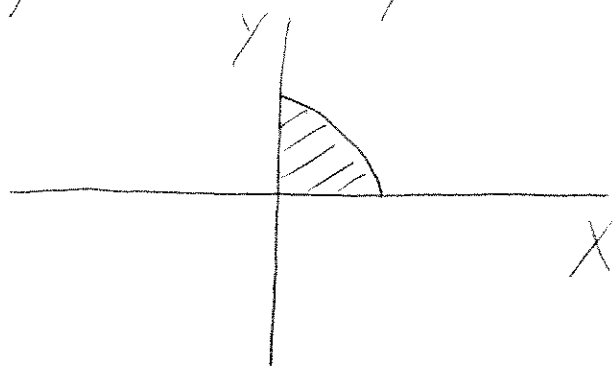
$$= 18\theta + 36\cos\theta + 9\theta - \frac{9}{2}\sin(2\theta)\Big|_0^{2\pi}$$

$$= (36\pi + 36 + 18\pi - 0) - (0 + 36 + 0 - 0) = \boxed{54\pi}$$

13.4.24 Evaluate by using polar coordinates. Sketch the region of integration first.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sin\left(x^2+y^2\right) dx\,dy$$

Region of integration:



Converting to polar:

$$\int_0^{\pi/2} \int_0^1 \sin(r^2)\, r\, dr\, d\theta \qquad u = r^2 \qquad \frac{du}{2} = r\,dr$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^1 \sin(u)\, du\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(-\cos(u) \Big|_0^1 \right) d\theta$$

$$= \frac{1-\cos(1)}{2} \int_0^{\pi/2} d\theta = \boxed{\frac{\pi}{4}\left(1-\cos(1)\right)}$$

13.4.37 Show that

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}dydx = \frac{\pi}{4}$$

Converting to polar we get:

$$\int_0^{\pi/2}\int_0^\infty \frac{r}{(1+r^2)^2}\, dr\, d\theta$$

$u = 1+r^2$
$du = 2r\,dr$

$$= \frac{1}{2}\int_0^{\pi/2}\int_1^\infty \frac{du}{u^2}\, dr\, d\theta$$

$$\cancel{= \frac{1}{2}\int_0^{\pi/2} \ln(u) \Big|_{u=1}^{u=\infty}}$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(-\frac{1}{u}\Big|_{u=1}^{u=\infty}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$$