Math 2210 - Assignment 8

Dylan Zwick

Fall 2008

1 Sections 12.9 through 13.1

1.1 Section 12.9

12.9.1 Find the minimum of:

$$f(x, y) = x^2 + y^2$$

subject to the constraint:

$$g(x, y) = xy - 3 = 0$$

$\nabla f(x,y) = \langle 2x, 2y \rangle \Rightarrow 2x = \lambda y$

$\nabla g(x,y) = \langle y, x \rangle \qquad 2y = \lambda x$

$xy = 3$

$\rightarrow y = \frac{2x}{\lambda} \Rightarrow 2\left(\frac{2x}{\lambda}\right) = \lambda x \Rightarrow \lambda = \pm 2$

$2x = \pm 2y \Rightarrow x = \pm y$

So, plugging into the constraint $\pm y^2 = 3$

We must take $+$, and $y = \sqrt{3}$, and so $x = \sqrt{3}$ and

$f_{max}(x,y) = f(\sqrt{3}, \sqrt{3}) = (\sqrt{3})^2 + (\sqrt{3})^2 = \boxed{6}$

12.9.4 Find the minimum of:

$$f(x, y) = x^2 + 4xy + y^2$$

subject to the constraint:

$$x - y - 6 = 0.$$

$\nabla f(x,y) = \langle 2x+4y,\ 2y+4x \rangle$

$\nabla g(x,y) = \langle 1, -1 \rangle$

$\Rightarrow \quad 2x + 4y = \lambda \qquad x - y - 6 = 0$

$\qquad 2y + 4x = -\lambda$

So, $\quad x = y + 6$ and so

$2(y+6) + 4y = \lambda \quad \Rightarrow \quad 6y + 12 = \lambda$

$2y + 4(y+6) = -\lambda \qquad\quad 6y + 24 = -\lambda$

$\Rightarrow \quad 12 = -2\lambda \Rightarrow \lambda = -6$

So, $\quad 6y + 12 = -6 \Rightarrow 6y = -18 \Rightarrow y = -3$

$\Rightarrow \quad x = y + 6 = -3 + 6 = 3$

So, $(3, -3)$ is our point and the minimum is:

$f(3,-3) = 3^2 + 4(3)(-3) + (-3)^2 = 18 - 36 = \boxed{-18}$

12.9.6 Find the minimum of:

$$f(x, y, z) = 4x - 2y + 3z$$

subject to the constraint:

$$2x^2 + y^2 - 3z = 0.$$

$\nabla f(x,y,z) = \langle 4, -2, 3 \rangle$

$\nabla g(x,y,z) = \langle 4x, 2y, -3 \rangle$

$4 = 4\lambda x$
$-2 = 2\lambda y$
$3 = -3\lambda$

$\Rightarrow$ $\lambda = -1$
$4 = -4x$
$-2 = -2y$

$\Rightarrow$ $x = -1$
$y = 1$

$2(-1)^2 + 1^2 - 3z = 0 \Rightarrow 3 - 3z = 0.$

So, $z = 1$.

Thus, our critical point is $(-1, 1, 1)$

$f(-1, 1, 1) = 4(-1) - 2(1) + 3(1)$
$= \boxed{-3}$

12.9.8 Find the minimum distance between the origin and the plane:

$$x + 3y - 2z = 4$$

$$d = \sqrt{x^2 + y^2 + z^2}$$

If the distance is minimized, so is the square of the distance

$$d^2 = x^2 + y^2 + z^2$$

$$\nabla d^2(x, y, z) = \langle 2x, 2y, 2z \rangle$$

$$\nabla g(x, y, z) = \langle 1, 3, -2 \rangle$$

$$2x = \lambda \qquad \Rightarrow 2y = 6x \qquad \Rightarrow 3y = \frac{18}{2}x = 9x$$

$$2y = 3\lambda \qquad 2z = -4x$$

$$2z = -2\lambda$$

So,

$$\cancel{x + \frac{9}{2}x + 4x = 4 \Rightarrow \frac{19}{2}x = 4 \Rightarrow x = \frac{8}{19}}$$

$$\cancel{x = \frac{8}{19}} \qquad x + 9x + 4x = 4 \Rightarrow 14x = 4$$

$$\cancel{y = \frac{24}{19}} \qquad \cancel{x = \frac{2}{7}} \qquad x = 2/7$$

$$\cancel{y} \qquad y = 6/7$$

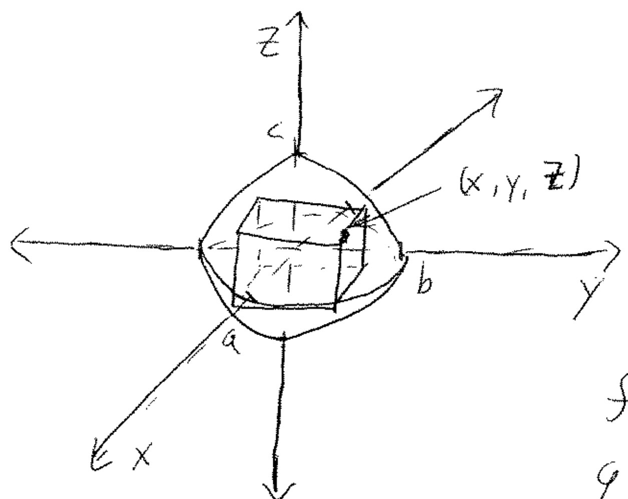
$$\cancel{z = \frac{16}{19}} \qquad z = -4/7$$

The minimum distance is then:

$$d\left(\frac{2}{7}, \frac{6}{7}, -\frac{4}{7}\right) = \frac{1}{7}\sqrt{4 + 36 + 16} = \frac{\sqrt{56}}{7} = \boxed{\frac{2\sqrt{14}}{7}}$$

12.9.11 Find the maximum volume of a closed rectangular box with faces parallel to the coordinate planes inscribed in the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



The volume will be:

$(2x)(2y)(2z) = 8xyz$

So,

$f(x,y,z) = 8xyz$

$g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

$\nabla f(x,y,z) = \langle 8yz, 8xz, 8xy \rangle$

$\nabla g(x,y,z) = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$

$8yz = \frac{2\lambda x}{a^2} \Rightarrow \lambda = \frac{4a^2yz}{x}$

$8xz = \frac{2\lambda y}{b^2} \Rightarrow xz = \frac{a^2y^2z}{xb^2} \Rightarrow x^2 = \frac{a^2y^2}{b^2}$

$8xy = \frac{2\lambda z}{c^2} \Rightarrow xy = \frac{a^2yz^2}{xc^2} \Rightarrow x^2 = \frac{a^2z^2}{c^2}$

So, $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

So, $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = 1/3 \Rightarrow x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$

And the maximal volume is:

$V = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \boxed{\frac{8abc}{3\sqrt{3}}}$

1.2 Section 13.1

13.1.1 Let $R = \{(x, y) : 1 \leq x \leq 4, 0 \leq y \leq 2\}$. Evaluate $\displaystyle\int\int_R f(x, y)dA$, where f is the function:

$$f(x, y) = \begin{cases} 2 & 1 \leq x < 3, 0 \leq y \leq 2 \\ 3 & 3 \leq x \leq 4, 0 \leq y \leq 2 \end{cases}$$

$$\int\int_R f(x,y)\,dA = 2(2)(2) + 3(1)(2)$$

$$= 8 + 6$$

$$= \boxed{14}$$

13.1.4 Let $R = \{(x, y) : 1 \le x \le 4, 0 \le y \le 2\}$. Evaluate $\displaystyle\int\int_R f(x, y)\,dA$, where f is the function:

$$f(x, y) = \begin{cases} 2 & 1 \le x \le 4, 0 \le y < 1 \\ 3 & 1 \le x < 3, 1 \le y \le 2 \\ 1 & 3 \le x \le 4, 1 \le y \le 2 \end{cases}$$

$$\iint_R f(x,y)\,dA = 2(4-1)(1-0) + 3(3-1)(2-1) + 1(4-3)(2-1)$$

$$= 2(3)(1) + 3(2)(1) + 1(1)(1)$$

$$= 6 + 6 + 1$$

$$= \boxed{13}$$

13.1.6 Suppose that

$$R = \{(x, y) : 0 \le x \le 2, 0 \le y \le 2\}$$
$$R_1 = \{(x, y) : 0 \le x \le 2, 0 \le y \le 1\}$$
$$R_2 = \{(x, y) : 0 \le x \le 2, 1 \le y \le 2\}.$$

Suppose, in addition, that:

$$\int\int_R f(x, y)dA = 3, \int\int_R g(x, y)dA = 5, \text{ and } \int\int_{R_1} g(x, y)dA = 2.$$

Use the properties of integrals to evaluate the integral:

$$\int\int_R [2f(x, y) + 5g(x, y)]dA$$

$$\iint_R [2f(x,y) + 5g(x,y)]\,dA$$

$$= 2\iint_R f(x,y)\,dA + 5\iint_R g(x,y)\,dA$$

$$= 2(3) + 5(5) = \boxed{31}$$

13.1.8 Suppose that

$$R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2\}$$
$$R_1 = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$$
$$R_2 = \{(x, y) : 0 \leq x \leq 2, 1 \leq y \leq 2\}.$$

Suppose, in addition, that:

$$\int\int_R f(x, y)dA = 3, \int\int_R g(x, y)dA = 5, \text{ and } \int\int_{R_1} g(x, y)dA = 2.$$

Use the properties of integrals to evaluate the integral:

$$\int\int_{R_1} [2g(x, y) + 3]dA$$

$$\iint_{R_1} [2g(x,y) + 3]dA = 2\iint_{R_1} g(x,y)dA + 3\iint_{R_1} dA$$

$$= 2(2) + 3\left((2-0)(1-0)\right)$$

$$= 4 + 6 = \boxed{10}$$

13.1.11 Suppose

$$R = \{(x, y) : 0 \le x \le 6, 0 \le y \le 4\}$$

and P is the partition of R into six equal squares by the lines:

$$x = 2, x = 4, \text{ and } y = 2.$$

Approximate $\int\int_R f(x, y)dA$ by calculating the corresponding Riemann sum $\sum_{k=1}^{6} f(\overline{x}_k, \overline{y}_k)\Delta A_k$, assuming that $(\overline{x}_k, \overline{y}_k)$ are the centers of the six squares.

$$f(x, y) = x^2 + 2y^2$$

$f(1,1) = 3$ $\quad f(3,1) = 11$ $\quad f(5,1) = 27$

$f(1,3) = 19$ $\quad f(3,3) = 27$ $\quad f(5,3) = 43$

$$\sum_{k=1}^{6} f(\overline{x}_k, \overline{y}_k)\Delta A_k$$

$$= (3+11+27+19+27+43)\,4$$

$$= 130(4) = \boxed{520}$$