Math 2210 - Assignment 7

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1 Sections 12.7 through 12.8

1.1 Section 12.7

12.7.1 Find the equation of the tangent plane to the surface:

$$x^2 + y^2 + z^2 = 16$$

at the point $(2, 3, \sqrt{3})$.

$F(x,y,z) = x^2 + y^2 + z^2$

$\nabla F(x,y,z) = \langle 2x, 2y, 2z \rangle$

$\nabla F(2,3,\sqrt{3}) = \langle 4, 6, 2\sqrt{3} \rangle$

$4(x-2) + 6(y-3) + 2\sqrt{3}(z - \sqrt{3}) = 0$

$\Rightarrow 4x + 6y + 2\sqrt{3}z = 32$

$\Rightarrow \boxed{2x + 3y + \sqrt{3}z = 16}$

12.7.4 Find the equation of the tangent plane to the surface:

$$x^2 + y^2 - z^2 = 4$$

at the point $(2, 1, 1)$

$$F(x,y,z) = x^2 + y^2 - z^2 = 4$$

$$\nabla F(x,y,z) = \langle 2x, 2y, -2z \rangle$$

$$\nabla F(2,1,1) = \langle 4, 2, -2 \rangle$$

The equation for the plane will be:

$$4(x-2) + 2(y-1) + (-2)(z-1) = 0$$

$$4x - 8 + 2y - 2 - 2z + 2 = 0$$

$$\Rightarrow 4x + 2y - 2z = 8$$

$$\Rightarrow \boxed{2x + y - z = 4}$$

12.7.7 Find the equation of the tangent plane to the surface:

$$z = 2e^{3y}\cos(2x)$$

at the point $(\pi/3, 0, -1)$.

The tangent plane has the equation

$$z = 2e^{3(0)}\cos\left(\frac{\pi}{3}\cdot 2\right) + \left(-4e^{3\cancel{0}}\sin(2\cancel{\pi/3})\right)\left(x-\frac{\pi}{3}\right) + 6e^{3\cancel{0}}\cos(2\cancel{\pi/3})(y-0)$$

$$z = 2\left(-\frac{1}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right)\left(x-\frac{\pi}{3}\right) + 6\left(-\frac{1}{2}\right)(y-0)$$

$$z = -1 - 2\sqrt{3}\,x + \frac{2\sqrt{3}}{3}\pi - 3y$$

$$\Rightarrow \boxed{2\sqrt{3}\,x + 3y + z = \left(\frac{2}{\sqrt{3}}\pi - 1\right)}$$

12.7.11 Use the total differential dz to approximate the change in

$$z = \ln(x^2 y)$$

as (x, y) moves from $(-2, 4)$ to $(-1.98, 3.96)$.

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$\frac{\partial z}{\partial x} = \frac{2xy}{x^2 y} = \frac{2}{x}$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{x^2 y} = \frac{1}{y}$$

$$dz = \left(\frac{2}{-2}\right)(.02) + \left(\frac{1}{4}\right)(-.04)$$

$$= \boxed{-.02}$$

12.7.29 For the function

$$f(x,y) = \sqrt{x^2 + y^2},$$

find the second-order Taylor approximation based at $(x_0, y_0) = (3, 4)$. Then estimate $f(3.1, 3.9)$ using

1. the first-order approximation,
2. the second-order approximation,
3. your calculator directly.

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_{xx}(x,y) = \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$f_{yy}(x,y) = \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$f_{xy}(x,y) = \frac{-xy}{(x^2+y^2)^{3/2}}$$

1. $f(3.1, 3.9) \approx \sqrt{3^2+4^2} + \frac{3}{\sqrt{3^2+4^2}}(3.1-3) + \frac{4}{\sqrt{3^2+4^2}}(3.9-4)$

$= 5 + \frac{3}{5}(.1) - \frac{4}{5}(.1) = 4.98$

2. $f(3.1, 3.9) \approx 4.98 + \frac{1}{2}\left(\frac{16}{\cancel{5\sqrt{5}}\ 125}(.1)^2 + \frac{24}{\cancel{5\sqrt{5}}\ 125}(.1)^2 + \frac{9}{\cancel{5\sqrt{5}}\ 125}(.1)^2\right)$

$= \cancel{5.073}$ 4.98196

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3. $f(3.1, 3.9) = 4.9819679$

1.2 Section 12.8

12.8.1 For the function

$$f(x, y) = x^2 + 4y^2 - 4x$$

find all critical points. Indicate whether each such point gives a local maximum or a local minimum, or whether it is a saddle point.

It's a polynomial, so its differentiable everywhere, and the domain is all of $\mathbb{R}^2$, and so the only critical points are stationary points.

$$\nabla f(x,y) = \langle 2x-4, 8y \rangle$$

$\nabla f(x,y) = \vec{0}$ when $x=2$, $y=0$

$f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 8$, $f_{xy}(x,y) = 0$

$D(2,0) = 16$ $\quad f_{xx}(2,0) = 2 > 0$

$\Rightarrow$ $\boxed{(2,0) \text{ is a local minimum}}$

12.8.6 For the function

$$f(x, y) = x^3 + y^3 - 6xy$$

find all critical points. Indicate whether each such point gives a local maximum or a local minimum, or whether it is a saddle point.

It's a polynomial, so its differentiable everywhere, and the domain is all of $\mathbb{R}^2$, and so the only critical points are stationary points.

$\nabla f(x,y) = \langle 3x^2 - 6y,\ 3y^2 - 6x \rangle$

$3x^2 - 6y = 0 \Rightarrow y = \frac{x^2}{2}$

$3y^2 - 6x = 0 \Rightarrow 3\left(\frac{x^2}{2}\right)^2 - 6x = 0$

$\Rightarrow \frac{3x^4}{4} - 6x = 0 \qquad x = 0$

or, $\frac{3x^3}{4} - 6 = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$

if $x = 0$, $y = 0$ if $x = 2$, $y = 2$

$f_{xx}(x,y) = 6x \quad f_{yy}(x,y) = 6y \Rightarrow D(x,y) = 36xy - 36$

$f_{xy}(x,y) = -6$

$D(0,0) = -36 \Rightarrow (0,0)$ is a saddle point.

$D(2,2) = 108$, $f_{xx}(2,2) = 12 \Rightarrow (2,2)$ is a local min.

12.8.15 Express a positive number N as a sum of three positive numbers such that the product of these three numbers is a maximum.

$$N = x + y + z$$

$$f(x,y,z) = xyz \qquad z = N - x - y$$

$$\Rightarrow f(x,y) = xy(N-x-y) = Nxy - x^2y - xy^2$$

$$f_x(x,y) = Ny - 2xy - y^2$$

$$f_y(x,y) = Nx - x^2 - 2xy$$

$$f_{xx}(x,y) = -2y \qquad f_{yy}(x,y) = -2x \qquad f_{xy}(x,y) = N - 2x - 2y$$

$$f_x(x,y) = 0 \Rightarrow 2xy = Ny - y^2 \Rightarrow x = \frac{N}{2} - \frac{y}{2}$$

$$f_y(x,y) = 0 \Rightarrow Nx - x^2 - 2xy = 0$$

$$\Rightarrow N\left(\frac{N}{2} - \frac{y}{2}\right) - \left(\frac{N}{2} - \frac{y}{2}\right)^2 - 2\left(\frac{N}{2} - \frac{y}{2}\right)y = 0$$

$$\frac{N^2}{2} - \frac{Ny}{2} - \frac{N^2}{4} + \frac{Ny}{2} - \frac{y^2}{4} - Ny + y^2 = 0$$

$$\Rightarrow \frac{3}{4}y^2 - Ny + \frac{N^2}{4} = 0 \Rightarrow 3y^2 - 4Ny + N^2 = 0$$

$$\Rightarrow (3y - N)(y - N) \Rightarrow y = \frac{N}{3}$$ (Note $y = N$ would not be allowed)

$$x = \frac{N}{2} - \frac{N}{3}/2 = \frac{N}{3}$$

$$f_{xy}(x,y) = -2y$$

$$f_{yy}(x,y) = -2x$$

$$f_{xy}(x,y) = N - 2x - 2y$$

$$D(x,y) = 4xy - (N-2x-2y)^2 \qquad f_{xy}\left(\frac{N}{3}, \frac{N}{3}\right) = -\frac{2N}{3} < 0$$

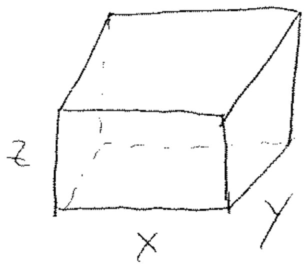
$$D\left(\frac{N}{3}, \frac{N}{3}\right) = \frac{4N^2}{9} - \left(-\frac{N}{3}\right)^2 = \frac{3N^2}{9} = \frac{N^2}{3} > 0$$

$\Rightarrow \left(\frac{N}{3}, \frac{N}{3}\right)$ is a local ~~min.~~ max

$$z = N - \frac{N}{3} - \frac{N}{3} = \frac{N}{3}$$

So, $\left(\frac{N}{3}, \frac{N}{3}, \frac{N}{3}\right)$ is ~~a~~ the ~~minimum~~ max, with product $\boxed{\frac{N^3}{27}}$

12.8.19 A rectangular metal tank with open top is to hold 256 cubic feet of liquid. What are the dimensions of the tank that require the least material to build.



$V = 256 = xyz$

$S = 2xz + 2yz + xy$

$z = \dfrac{256}{xy}$

$\Rightarrow S(x,y) = 2x\left(\dfrac{256}{xy}\right) + 2y\left(\dfrac{256}{xy}\right) + xy$

$= \dfrac{512}{y} + \dfrac{512}{x} + xy$

$S_x(x,y) = y - \dfrac{512}{x^2} \qquad S_y(x,y) = x - \dfrac{512}{y^2}$

$S_{xx}(x,y) = \dfrac{1024}{x^3} \qquad S_{yy}(x,y) = \dfrac{1024}{y^3} \qquad S_{xy}(x,y) = 1$

$S_x(x,y) = 0 \Rightarrow y = \dfrac{512}{x^2} \Rightarrow S_y(x,y) = 0 \Rightarrow x - \dfrac{x^4}{512} = 0$

$\Rightarrow x = 0$ or $x^3 = 512 \Rightarrow x = 8$ $\quad x = 0$ is not allowed.

So, $x = 8, \Rightarrow y = 8$ $\qquad z = \dfrac{256}{8 \times 8} = 4$

$D(x,y) = \dfrac{(1024)^2}{x^3 y^3} - 1^2 = 3 > 0$

$\Rightarrow$ $\boxed{x = 8\text{ ft},\ y = 8\text{ ft},\ z = 4\text{ ft} \text{ will minimize the surface area with } 192\text{ ft}^2}$

$S_{xx}(8,8) = 2 > 0$

$\Rightarrow (8,8)$ is a minimum.

12.8.24 Find the minimum distance between the point (1, 2, 0) and the quadric cone

$$z^2 = x^2 + y^2$$

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2}$$

on the quadric cone we have

$$z^2 = x^2 + y^2$$

and d is minimized when d^2 is, so we get

$$d^2(x,y) = (x-1)^2 + (y-2)^2 + x^2 + y^2$$
$$= 2x^2 - 2x + 2y^2 - 4y + 5$$

$d^2_x(x,y) = 4x - 2$ $\quad d^2_y(x,y) = 4y - 4$ $\quad d^2_{xx}(x,y) = 4$ $\quad d^2_{yy}(x,y) = 4$

$d^2_{xy}(x,y) = 0$

$\Rightarrow d^2_x(x,y) = 0 \Rightarrow x = \frac{1}{2}$ $\quad d^2_y(x,y) = 0 \Rightarrow y = 1$

So, $(\frac{1}{2}, 1)$ is the stationary point

$D(x,y) = 16$, $\quad d^2_{xx}(x,y) = 4 > 0$ everywhere.

So, $(\frac{1}{2}, 1)$ is a minimum.

The minimum distance occurs at $\left(\frac{1}{2}, 1, \pm\frac{\sqrt{5}}{2}\right)$

and its value is:

$$\sqrt{\left(\tfrac{1}{2}-1\right)^2 + (1-2)^2 + \left(\tfrac{\sqrt{5}}{2}\right)^2} = \sqrt{\tfrac{1}{4} + 1 + \tfrac{5}{4}} = \boxed{\frac{\sqrt{10}}{2}}$$