Math 2210 - Assignment 6

Dylan Zwick

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1 Sections 12.4 through 12.6

1.1 Section 12.4

12.4.1 Find the gradient, $\nabla f(x, y)$, of the function $f(x, y)$:

$$f(x, y) = x^2y + 3xy$$

$\nabla f(x,y) = \langle 2xy + 3y, \; x^2 + 3x \rangle$

12.4.3 Find the gradient, $\nabla f(x, y)$, of the function $f(x, y)$:

$$f(x, y) = xe^{xy}$$

$\nabla f(x,y) = \langle xye^{xy} + e^{xy},\ x^2 e^{xy} \rangle$

12.4.8 Find the gradient, $\cancel{\nabla f(x, y)}$ $\nabla f(x, y, z)$, of the function $\cancel{f(x, y)}$ $f(x, y, z)$:

$$\cancel{f(x, y)}\ f(x, y, z) = x^2y + y^2z + z^2x$$

$\cancel{\nabla f(x, y) = \langle}$

$\nabla f(x, y, z) = \langle 2xy + z^2,\ x^2 + 2yz,\ y^2 + 2zx \rangle$

12.4.11 Find the gradient vector of the given function at the given point $\mathbf{p}$. Then find the equation of the tangent plane at $\mathbf{p}$.

$$f(x,y) = x^2y - xy^2,\ \mathbf{p} = (-2,3)$$

$$\nabla f(x,y) = \langle 2xy - y^2,\ x^2 - 2xy \rangle$$

The tangent plane is.

$$z = f(-2,3) + f_x(-2,3)(x-(-2)) + f_y(-2,3)(y-3)$$

$$f(-2,3) = (-2)^2(3) - (-2)(3^2) = 12 + 18 = 30$$

$$f_x(-2,3) = -21 \qquad f_y(-2,3) = 16$$

$$\Rightarrow \quad z = 30 - 21(x+2) + 16(y-3)$$

$$\Rightarrow \quad z = 30 - 21x - 42 + 16y - 48$$

$$\Rightarrow \quad \boxed{z = -21x + 16y - 60}$$

or

$$\boxed{21x - 16y + z = -60}$$

12.4.20 Find all points (x, y) at which the tangent plane to the graph of $z = x^3$ is horizontal.

The tangent plane has equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

which is horizontal if and only if

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

$f_x = 3x^2 \quad f_y = 0$

$\Rightarrow \quad x = 0.$

So, for all points $(0, y, 0)$ on the surface $z = x^3$ the tangent plane is horizontal.

1.2 Section 12.5

12.5.1 Find the directional derivative of f at the point $\mathbf{p}$ in the direction of $\mathbf{a}$:

$$f(x, y) = x^2 y;\ \mathbf{p} = (1, 2);\ \mathbf{a} = 3\mathbf{i} - 4\mathbf{j}.$$

$$f_x(x,y) = 2xy$$

$$f_y(x,y) = x^2$$

$$\nabla f(1,2) = \langle 4, 1 \rangle$$

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$D_{\vec{u}} f(1,2) = \langle 4, 1 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

$$= \frac{12}{5} - \frac{4}{5} = \boxed{\frac{8}{5}}$$

12.5.6 Find the directional derivative of f at the point $\mathbf{p}$ in the direction of $\mathbf{a}$:

$$f(x, y) = e^{-xy};\ \mathbf{p} = (1, -1);\ \mathbf{a} = -\mathbf{i} + \sqrt{3}\mathbf{j}.$$

$$f_x(x,y) = -y\,e^{-xy}$$

$$f_y(x,y) = -x\,e^{-xy}$$

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

$$\nabla f(1,-1) = \langle e, -e \rangle$$

$$D_{\vec{a}} f(1,-1) = \langle e, -e \rangle \cdot \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= \boxed{-e\left(\frac{1+\sqrt{3}}{2}\right)}$$

12.5.8 Find the directional derivative of f at the point $\mathbf{p}$ in the direction of $\mathbf{a}$:

$$f(x, y, z) = x^2 + y^2 + z^2; \mathbf{p} = (1, -1, 2); \mathbf{a} = \sqrt{2}\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

$f_x(x, y, z) = 2x$

$f_y(x, y, z) = 2y$

$f_z(x, y, z) = 2z$

$\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{\sqrt{2}}{2}\hat{i} - \dfrac{1}{2}\hat{j} - \dfrac{1}{2}\hat{k}$

$$D_{\vec{a}} f(1, -1, 2) = \langle 2, -2, 4 \rangle \cdot \left\langle \frac{\sqrt{2}}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$= \sqrt{2} + 1 - 2$$

$$= \boxed{\sqrt{2} - 1}$$

12.5.14 In what direction **u** does $f(x, y) = \sin(3x - y)$ decrease most rapidly at $\mathbf{p} = (\pi/6, \pi/4)$.

$f_x(x, y) = 3\cos(3x - y)$

$f_y(x, y) = -\cos(3x - y)$

$\cos\left(3\left(\pi/6\right) - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\nabla f(\pi/6, \pi/4) = \left\langle \frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

The gradient points in the direction of maximal increase, so:

$$\vec{u} = \frac{\left\langle \frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle}{\cancel{\left(\frac{5}{2}\right)}\sqrt{5}} = \left\langle \frac{3}{\sqrt{5}\sqrt{2}}, -\frac{1}{\sqrt{5}\sqrt{2}} \right\rangle$$

$$= \boxed{\left\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle}$$

12.5.21 Find the gradient of $f(x, y, z) = \sin\sqrt{x^2 + y^2 + z^2}$. Show that the gradient always points directly toward the origin or directly away from the origin.

$$\nabla f(x, y, z) = \frac{\cos\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}} \langle x, y, z\rangle$$

Now, if $(x, y, z) = (0, 0, 0)$ ~~here~~ then the gradient is not well defined.

If $(x, y, z) \neq (0, 0, 0)$ then if $\cos\sqrt{x^2+y^2+z^2} = 0$ it points in no direction at all.

However, in any other case the gradient is either parallel or antiparallel to the vector $\langle x, y, z\rangle$, which points directly away from the origin.

1.3 Section 12.6

12.6.1 Find dw/dt by using the chain rule. Express your final answer in terms of t.

$$w = x^2y^3; x = t^3, y = t^2.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = 2xy^3 = 2(t^3)(t^2)^3 = 2t^9$$

$$\frac{dy}{dt} = 3t^2$$

$$\frac{\partial w}{\partial y} = 3x^2y^2 = 3(t^3)^2(t^2)^2 = 3t^{10}$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dw}{dt} = 6t^{11} + 6t^{11} = \boxed{12t^{11}}$$

12.6.4 Find dw/dt by using the chain rule. Express your final answer in terms of t.

$$w = \ln(x/y);\ x = \tan t,\ y = (\sec t)^2.$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dy}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{1}{\left(\frac{x}{y}\right)} \cdot \left(\frac{1}{y}\right) = \frac{1}{x} = \frac{1}{\tan(t)}$$

$$\frac{dx}{dt} = \sec^2(t)$$

$$\frac{\partial w}{\partial y} = \frac{1}{\left(\frac{x}{y}\right)} \cdot \left(-\frac{x}{y^2}\right) = -\frac{1}{y} = -\frac{1}{\sec^2(t)}$$

$$\frac{dy}{dt} = 2\sec^2 t \tan t$$

$$\Rightarrow \frac{dw}{dt} = \frac{1}{\tan(t)} \cdot \sec^2(t) - 2\tan(t)$$

$$= \frac{\sec^2(t) - 2\tan^2(t)}{\tan(t)} = \boxed{\frac{1 - \tan^2(t)}{\tan(t)}}$$

$$\sec^2(t) - \tan^2(t)$$

$$= \frac{1}{\cos^2(t)} - \frac{\sin^2(t)}{\cos^2(t)} = 1$$

$$\tan^2(t) + 1 = \sec^2(t)$$

12.6.7 Find $\partial w/\partial t$ by using the chain rule. Express your final answer in terms of s and t.

$$w = x^2y; x = st, y = s - t.$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2xy)(s) + x^2(-1)$$

$$= 2(st)(s-t)(s) + s^2t^2(-1)$$

$$= 2s^3t - 2s^2t^2 - s^2t^2$$

$$= \boxed{2s^3t - 3s^2t^2}$$

$$w(s,t) = s^3t^2 - s^2t^3$$

So, it checks out.

12.6.11 Find $\partial w/\partial t$ by using the chain rule. Express your final answer in terms of s and t.

$$w = \sqrt{x^2 + y^2 + z^2};\ x = \cos(st),\ y = \sin(st),\ z = s^2t.$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

in terms of s and t:

$$\frac{\partial w}{\partial x} = \frac{\cos(st)}{\sqrt{1+s^4t^2}} \qquad \frac{\partial w}{\partial y} = \frac{\sin(st)}{\sqrt{1+s^4t^2}} \qquad \frac{\partial w}{\partial z} = \frac{s^2t}{\sqrt{1+s^4t^2}}$$

$$\frac{\partial x}{\partial t} = -s\sin(st) \quad \frac{\partial y}{\partial t} = s\cos(st),\ \frac{\partial z}{\partial t} = s^2$$

$$\frac{\partial w}{\partial t} = -\frac{s\cos(st)\sin(st)}{\sqrt{1+s^4t^2}} + \frac{s\cos(st)\sin(st)}{\sqrt{1+s^4t^2}} + \frac{s^4t}{\sqrt{1+s^4t^2}}$$

$$= \boxed{\frac{s^4t}{\sqrt{1+s^4t^2}}}$$

12.6.20 Sand is pouring onto a conical pile in such a way that at a certain instant the height is 100 inches and increasing at 3 inches per minute and the base radius is 40 inches and increasing at 2 inches per minute. How fast is the volume increasing at that instant?

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r h\left(\frac{dr}{dt}\right) + \frac{1}{3}\pi r^2\left(\frac{dh}{dt}\right)$$

$$= \frac{2}{3}\pi (40)(100)(2) + \frac{1}{3}\pi (40)^2 (3)$$

$$= \frac{16{,}000\pi}{3} + \frac{4{,}800\pi}{3}$$

$$= \boxed{\frac{20{,}800}{3}\pi \ \text{in}^3/\text{min}}$$

$$\approx 21{,}782 \ \text{in}^3/\text{min}$$