Math 2210 - Assignment 5

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1 Sections 12.1 through 12.3

1.1 Section 12.1

12.1.1 Let $f(x, y) = x^2 y + \sqrt{y}$. Find each value.

1. $f(2, 1)$

$$f(2,1) = 2^2(1) + \sqrt{1}$$
$$= \boxed{5}$$

2. $f(3, 0)$

$$f(3,0) = 3^2(0) + \sqrt{0}$$
$$= \boxed{0}$$

3. $f(1, 4)$

$$f(1,4) = 1^2(4) + \sqrt{4}$$
$$= \boxed{6}$$

4. $f(a, a^4)$

$$f(a, a^4) = a^2(a^4) + \sqrt{a^4}$$
$$= a^6 + a^2$$

5. $f(1/x, x^4)$

$$f\left(\frac{1}{x}, x^4\right) = \left(\frac{1}{x}\right)^2 x^4 + \sqrt{x^4}$$
$$= 2x^2$$

6. $f(2, -4)$

$$f(2, -4) = 2^2(-4) + \sqrt{-4}$$
$$= \boxed{\text{Not real}}$$
$$= \boxed{-16 + 2i}$$

What is the natural domain for this function?

$$-\infty < x < \infty$$
$$0 \le y < \infty$$

12.1.6 Find $F(f(t), g(t))$ if $F(x, y) = e^x + y^2$ and $f(t) = \ln t^2$, $g(t) = e^{t/2}$.

$$F(f(t), g(t)) = e^{(\ln t^2)} + (e^{t/2})^2$$

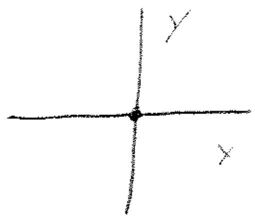
$$= \boxed{t^2 + e^t}$$

12.1.17 Sketch the level curve $z = k$ for the indicated values of k.

$$z = \frac{1}{2}(x^2 + y^2),\ k = 0, 2, 4, 6, 8.$$

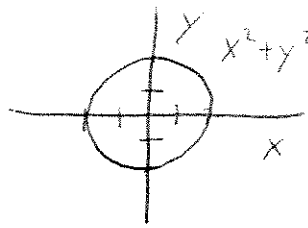
$k = 0$

$0 = \frac{1}{2}(x^2 + y^2)$



$k = 2$

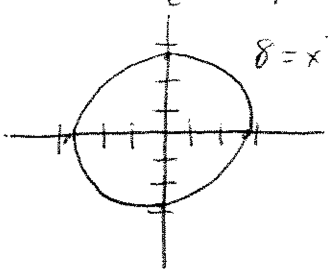
$2 = \frac{1}{2}(x^2 + y^2)$



$x^2 + y^2 = 4$

$k = 4$

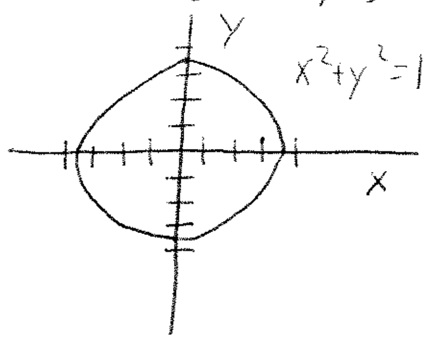
$4 = \frac{1}{2}(x^2 + y^2)$



$8 = x^2 + y^2$

$k = 6$

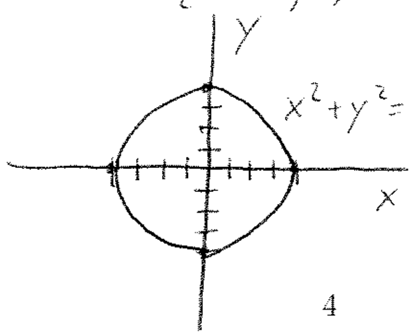
$6 = \frac{1}{2}(x^2 + y^2)$



$x^2 + y^2 = 12$

$k = 8$

$8 = \frac{1}{2}(x^2 + y^2)$



$x^2 + y^2 = 16$

12.1.27 Describe geometrically the domain of the function:

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 16}.$$

Domain is where

$$x^2+y^2+z^2-16 \geq 0$$

$$\Rightarrow x^2+y^2+z^2 \geq 16$$

The points outside the open ball of radius 4 around the origin.

12.1.33 Describe geometrically the level surfaces for the function:

$$f(x, y, z) = x^2 + y^2 + z^2; k > 0$$

$$x^2 + y^2 + z^2 = k$$

The level surfaces are spheres of radius $\sqrt{k}$ around the origin.

1.2 Section 12.2

12.2.1 Find all the partial derivatives of the function:

$$f(x, y) = (2x - y)^4$$

$$f_x(x,y) = 4(2x-y)^3(2)$$

$$= \boxed{8(2x-y)^3}$$

$$f_y(x,y) = 4(2x-y)^3(-1)$$

$$= \boxed{-4(2x-y)^3}$$

12.2.5 Find all the partial derivatives of the function:

$$f(x, y) = e^y \sin x$$

$$f_x(x,y) = e^y \cos x$$

$$f_y(x,y) = e^y \sin x$$

12.2.13 Find all the partial derivatives of the function:

$$f(x, y) = y\cos x^2 + y^2$$

$$\boxed{f_x(x,y) = -2xy\sin(x^2)}$$

$$\boxed{f_y(x,y) = \cos(x^2) + 2y}$$

12.2.19 Verify that:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

for the function:

$$f(x, y) = 3e^{2x} \cos y$$

$$\frac{\partial f}{\partial x} = 6e^{2x} \cos y \qquad \frac{\partial f}{\partial y} = -3e^{2x} \sin y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -6e^{2x} \sin y \qquad \frac{\partial^2 f}{\partial x \partial y} = -6e^{2x} \sin y$$

So,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

12.2.34 A function of two variables that satisfies *Laplace's Equation*,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is said to be *harmonic*. Show that the function:

$$\cancel{f(x,y) = \ln(4x^2 + 4^2)}$$

$$f(x,y) = \ln(4x^2 + 4y^2)$$

is harmonic.

$$\frac{\partial f}{\partial x} = \frac{8x}{4x^2 + 4y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(4x^2+4y^2)8 - 8x(8x)}{(4x^2+4y^2)^2} = \frac{32y^2 - 32x^2}{(4x^2+4y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{8y}{4x^2+4y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(4x^2+4y^2)8 - 8y(8y)}{(4x^2+4y^2)^2} = \frac{32x^2 - 32y^2}{(4x^2+4y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{32y^2 - 32x^2 + 32x^2 - 32y^2}{(4x^2+4y^2)^2} = \boxed{0}$$

1.3 Section 12.3

12.3.1 Find the limit or state that it does not exist:

$$\lim_{(x,y)\to(1,3)} (3x^2y - xy^3)$$

$3x^2y - xy^3$ is a polynomial, and so continuous everywhere.

Therefore,

$$\lim_{(x,y)\to(1,3)} (3x^2y - xy^3)$$

$$= 3(1^2)(3) - (1)(3^3)$$

$$= 9 - 27$$

$$= \boxed{-18}$$

12.3.4 Find the limit or state that it does not exist:

$$\lim_{(x,y)\to(1,2)} \frac{x^3 - 3x^2y + 3xy^2 - y^3}{y - 2x^2}$$

$$\frac{x^3 - 3x^2y + 3xy^2 - y^3}{y - 2x^2}$$

$$\frac{1^3 - 3(1^2)(2) + 3(1)(2^2) - 2^3}{2 - 2(1^2)}$$

$$= \frac{1 - 6 + 12 - 8}{0} = \frac{-1}{0}$$

$\Rightarrow$ undefined.

Does not exist.

12.3.11 Find the limit or state that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

Take limit along line $x=0$

$$\lim_{y\to 0} \frac{0}{y} = 0$$

take limit along line $x=y$

$$\lim_{x\to 0} \frac{x^2}{\sqrt{2x^2}} = 0.$$

So, looks promising. Convert to polar:

$$\lim_{r\to 0} \frac{r\cos\theta \, r\sin\theta}{r} = \lim_{r\to 0} r\cos\theta\sin\theta$$

$$-r < r\cos\theta\sin\theta < r$$

$$\lim_{r\to 0} -r = \lim_{r\to 0} r = 0.$$

So, $\lim_{r\to 0} r\cos\theta\sin\theta = 0.$

So, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

12.3.16 Find the limit or state that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

Take limit along line $y=x$

$$\lim_{x\to 0} \frac{x^3}{x^2+x^4} = 0$$

Take limit along line $y^2=x$

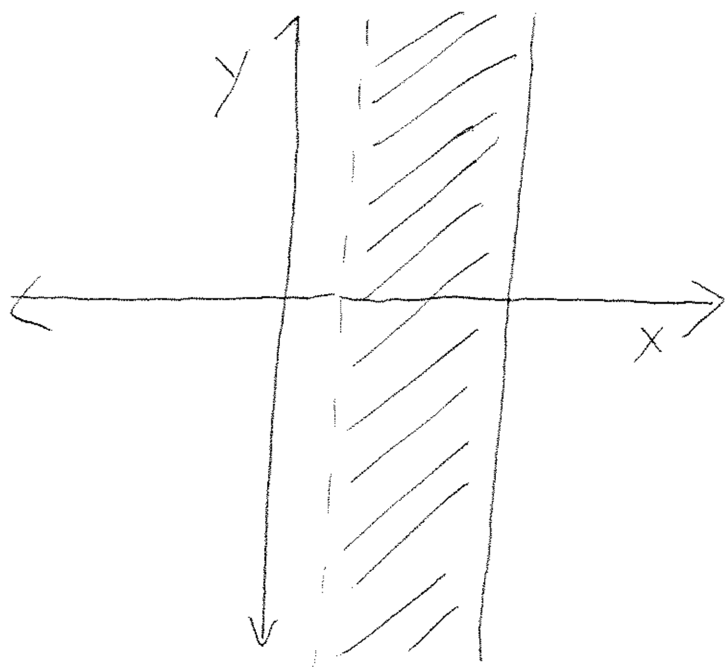
$$\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$$0 \neq \frac{1}{2}$$

So, the limit does not exist.

12.3.30 Sketch the set S and describe the boundary of the set. Finally, state whether the set is open, closed, or neither.

$$S = \{(x, y) : 1 < x \leq 4\}$$



Neither open nor closed.