Math 2210 - Assignment 3

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1 Sections 11.5 and 11.6

1.1 Section 11.5

11.5.3 Find the required limit or indicate that it does not exist.

$$\lim_{t \to 1}\left[\frac{t-1}{t^2-1}\mathbf{i} - \frac{t^2+2t-3}{t-1}\mathbf{j}\right]$$

$$\lim_{t \to 1} \frac{t-1}{t^2-1} = \lim_{t \to 1} \frac{\cancel{(t-1)}}{(t+1)\cancel{(t-1)}} = \lim_{t \to 1} \frac{1}{t+1} = \frac{1}{2}$$

$$\lim_{t \to 1} \frac{t^2+2t-3}{t-1} = \lim_{t \to 1} \frac{(t+3)\cancel{(t-1)}}{\cancel{(t-1)}} = \lim_{t \to 1} t+3 = 4$$

So,

$$\lim_{t \to 1}\left[\frac{t-1}{t^2-1}\hat{i} - \frac{t^2+2t-3}{t-1}\hat{j}\right] = \boxed{\frac{1}{2}\hat{i} - 4\hat{j}}$$

11.5.9 When no domain is given in the definition of a vector-valued function, it is to be understood that the domain is the set of all (real) scalars for which the rule for the function makes sense and gives real vectors (i.e., vectors with real components). Find the domain of each of the following vector-valued functions:

1. $\mathbf{r}(t) = \frac{2}{t-4}\mathbf{i} + \sqrt{3-t}\,\mathbf{j} + \ln|4-t|\,\mathbf{k}$

$\frac{2}{t-4}$ defined for $t \neq 4$

$\sqrt{3-t}$ defined (or at least real) for $t \leq 3$

$\ln|4-t|$ real for $t < 4$

$\Rightarrow$ Domain is $\boxed{t \leq 3}$

2. $\mathbf{r}(t) = \wp(t^2)\mathbf{i} - \sqrt{20-t}\,\mathbf{j} + 3\mathbf{k}$ ($\wp$ denotes the greatest integer functions. This is different than the symbol that the textbook uses, as I could not find how to reproduce that symbol using the typesetting program I'm using.)

$[\![t^2]\!]$ defined for all t

$-\sqrt{20-t}$ real for $t \leq 20$

3 real for all t.

So, the domain is $\boxed{t \leq 20}$

3. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{9-t^2}\,\mathbf{k}$.

$\cos t$ defined and real for all t

$\sin t$ defined and real for all t

$\sqrt{9-t^2}$ is real for $-3 \leq t \leq 3$

So, domain is $\boxed{-3 \leq t \leq 3}$

11.5.14 Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for each of the following:

1. $\mathbf{r}(t) = (e^t + e^{-t^2})\mathbf{i} + 2^t\mathbf{j} + t\mathbf{k}$

$$\vec{r}\,'(t) = \boxed{(e^t - 2te^{-t^2})\hat{i} + (\ln 2)\, 2^t \hat{j} + \hat{k}}$$

$$\vec{r}\,''(t) = (e^t - 2e^{-t^2} + 4t^2 e^{-t^2})\hat{i} + (\ln 2)^2 2^t \hat{j} + 0\hat{k}$$

$$= \boxed{(e^t + (4t^2 - 2)e^{-t^2})\hat{i} + (\ln 2)^2 2^t \hat{j} + 0\hat{k}}$$

2. $\mathbf{r}(t) = \tan 2t\,\mathbf{i} + \arctan t\,\mathbf{j}$.

$$\boxed{\vec{r}\,'(t) = 2\sec^2(2t)\,\hat{i} + \frac{1}{1+t^2}\,\hat{j}}$$

$$\boxed{\vec{r}\,''(t) = 8\sec(2t)\tan(2t)\,\hat{i} - \frac{2t}{(1+t^2)^2}\,\hat{j}}$$

11.5.29 Find the velocity **v**, acceleration **a**, and speed s at the indicated time $t = t_1$.

$$\mathbf{r}(t) = t\sin\pi t\,\mathbf{i} + t\cos\pi t\,\mathbf{j} + e^{-t}\mathbf{k};\ t_1 = 2.$$

$$\vec{v} = \vec{r}\,'(t) = (\pi t\cos(\pi t) + \sin(\pi t))\,\hat{i} + (\cos(\pi t) - \pi t\sin(\pi t))\,\hat{j} - e^{-t}\,\hat{k}$$

$$\vec{v}(2) = (\sin(2\pi) + 2\pi\cos(2\pi))\,\hat{i} + (\cos(2\pi) - 2\pi\sin(2\pi))\,\hat{j} - e^{-2}\,\hat{k}$$

$$= \boxed{2\pi\,\hat{i} + \hat{j} - e^{-2}\,\hat{k}}$$

$$\vec{a} = \vec{r}\,''(t) = (\pi\cos(\pi t) + \pi\cos(\pi t) - \pi^2 t\sin(\pi t))\,\hat{i} + (-\pi\sin(\pi t) - \pi\sin(\pi t) - \pi^2 t\cos(\pi t))\,\hat{j} + e^{-t}\,\hat{k}$$

$$= (2\pi\cos(\pi t) - \pi^2 t\sin(\pi t))\,\hat{i} - (2\pi\sin(\pi t) + \pi^2 t\cos(\pi t))\,\hat{j} + e^{-t}\,\hat{k}$$

$$\vec{a}(2) = \boxed{2\pi\,\hat{i} - 2\pi^2\,\hat{j} + e^{-2}\,\hat{k}}$$

$$s(t) = \|\vec{v}(t)\| = \sqrt{[\pi t\cos(\pi t) + \sin(\pi t)]^2 + [\cos(\pi t) - \pi t\sin(\pi t)]^2 + [e^{-t}]^2}$$

$$\Rightarrow s(2) = \sqrt{(2\pi)^2 + 1^2 + (e^{-2})^2} = \boxed{\sqrt{1 + 4\pi^2 + e^{-4}}}$$

11.5.36 Find the length of the curve with the given vector equation.

$$\mathbf{r}(t) = t^2\mathbf{i} - 2t^3\mathbf{j} + 6t^3\mathbf{k};\ 0 \le t \le 1.$$

$$\vec{r}\,'(t) = 2t\,\hat{i} - 6t^2\,\hat{j} + 18t^2\,\hat{k}$$

$$L = \int_0^1 \sqrt{\|\vec{r}\,'(t)\|^2}\,dt$$

$$= \int_0^1 \sqrt{(2t)^2 + (-6t^2)^2 + (18t^2)^2}\,dt$$

$$= \int_0^1 \sqrt{4t^2 + 360t^4}\,dt$$

$$= \int_0^1 2t\sqrt{1 + 90t^2}\,dt$$

$$u = 1 + 90t^2$$

$$\Rightarrow \frac{du}{90} = 2t\,dt$$

$$= \int_1^{91} \frac{\sqrt{u}}{90}\,du$$

$$= \left(\frac{u^{3/2}}{90}\right)\left(\frac{2}{3}\right)\Big|_1^{91} = \frac{u^{3/2}}{135}\Big|_1^{91}$$

$$= \boxed{\frac{91\sqrt{91} - 1}{135}} \approx 6.423$$

1.2 Section 11.6

11.6.1 Find the parametric equations of the line through the given pair of points:

$$(1, -2, 3) \text{ and } (4, 5, 6).$$

$$\vec{v} = \langle 4-1, 5-(-2), 6-3 \rangle$$

$$= \langle 3, 7, 3 \rangle$$

$$P = (1, -2, 3)$$

$$\boxed{\begin{aligned} x(t) &= 1+3t \\ y(t) &= -2+7t \\ z(t) &= 3+3t \end{aligned}}$$

11.6.7 Write both the parametric equations and the symmetric equations for the line through the given point parallel to the given vector:

$$(1, 1, 1), < -10, -100, -1000 >.$$

Parametric:

$$x(t) = 1 - 10t$$
$$y(t) = 1 - 100t$$
$$z(t) = 1 - 1000t$$

Symmetric:

$$\frac{x-1}{-10} = \frac{y-1}{-100} = \frac{z-1}{-1000}$$

equivalently

$$\boxed{\frac{x-1}{10} = \frac{y-1}{100} = \frac{z-1}{1000}}$$

11.6.11 Find the symmetric equations of the line of intersection of the given pair of planes.

$$x + 4y - 2z = 13$$

and

$$2x - y - 2z = 5$$

The normal vector to the first plane is:

$$\vec{n}_1 = \langle 1, 4, -2 \rangle$$

The normal vector to the second plane is:

$$\vec{n}_2 = \langle 2, -1, -2 \rangle$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & -2 \\ 2 & -1 & -2 \end{vmatrix} = (-8-2)\hat{i} + (-4-(-2))\hat{j} + (-1-8)\hat{k}$$

$$= -10\hat{i} - 2\hat{j} - 9\hat{k}$$

If we set $y = 0$ we can find a point on both planes:

$$\begin{aligned} x - 2z &= 13 \\ 2x - 2z &= 5 \end{aligned} \Rightarrow -x = 8 \Rightarrow x = -8$$

$$-2z = 21 \Rightarrow z = \frac{21}{2}$$

So, $(-8, 0, \frac{21}{2})$ is on the plane and we get the symmetric equations:

$$\boxed{\frac{x+8}{10} = \frac{y}{2} = \frac{z - \frac{21}{2}}{9}}$$

11.6.23 Find the symmetric equations of the tangent line to the curve with equation

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 6\sin t\,\mathbf{j} + t\mathbf{k}$$

at $t = \pi/3$

$$\vec{r}(\pi/3) = 2\cos\left(\tfrac{\pi}{3}\right)\hat{i} + 6\sin\left(\tfrac{\pi}{3}\right)\hat{j} + \left(\tfrac{\pi}{3}\right)\hat{k}$$
$$= 1\hat{i} + 3\sqrt{3}\,\hat{j} + \tfrac{\pi}{3}\hat{k}$$

This is a point $(1, 3\sqrt{3}, \frac{\pi}{3})$ on the line.

$$\vec{r}\,'(t) = -2\sin(t)\hat{i} + 6\cos t\,\hat{j} + \hat{k} \qquad t = \pi/3$$

$$\vec{r}\,'\left(\tfrac{\pi}{3}\right) = -2\sin\left(\tfrac{\pi}{3}\right)\hat{i} + 6\cos\left(\tfrac{\pi}{3}\right)\hat{j} + \hat{k}$$
$$= -\sqrt{3}\,\hat{i} + 3\hat{j} + 1\hat{k} \Rightarrow \text{parallel vector.}$$

So, the symmetric equations are:

$$\boxed{\frac{x-1}{-\sqrt{3}} = \frac{y-3\sqrt{3}}{3} = \frac{z-\frac{\pi}{3}}{1}}$$

11.6.25 Find the equations of the plane perpendicular to the curve:

$$x(t) = 3t,\ y(t) = 2t^2,\ z(t) = t^5$$

at $t = -1$.

$x'(t) = 3 \quad y'(t) = 4t \quad z'(t) = 5t^4$

$x'(-1) = 3 \quad y'(-1) = -4 \quad z'(-1) = 5$

$\vec{n} = \langle 3, -4, 5 \rangle$

$x(-1) = -3 \quad y(-1) = 2, \quad z(-1) = -1$

$P = (-3, 2, -1)$

So, the plane is:

$3(x+3) - 4(y-2) + 5(z+1) = 0$

$3x + 9 - 4y + 8 + 5z + 5 = 0$

$\Rightarrow \boxed{3x - 4y + 5z = -22}$