Math 2210 - Assignment 11

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Section 13.8

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13.8.1 Evaluate the integral:

$$\int_0^{2\pi}\int_0^3\int_0^{12} r\,dz\,dr\,d\theta$$

and describe the region of integration.

$$\int_0^{2\pi}\int_0^3\int_0^{12} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 12r\,dr\,d\theta = \int_0^{2\pi} 6r^2\Big|_0^3 d\theta$$

$$= \int_0^{2\pi} 54\,d\theta = 108\pi$$

13.8.8 Find the volume of the solid bounded above by the sphere:

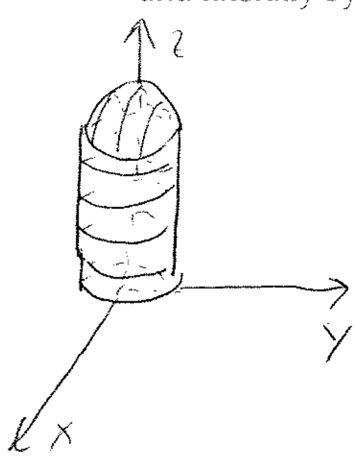
$$x^2 + y^2 + z^2 = 9,$$

below by the plane:

$$z = 0,$$

and laterally by the cylinder:

$$x^2 + y^2 = 4.$$



The volume will be:
(converting to cylindrical)

$$\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta$$

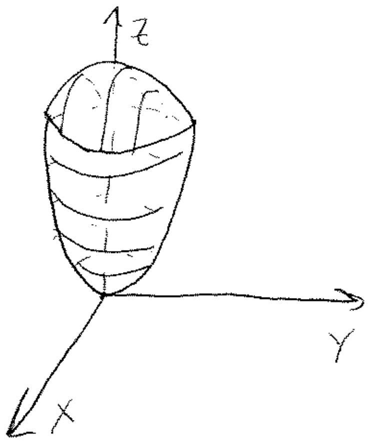
$$= \int_0^{2\pi} \int_0^2 r\sqrt{9-r^2} \, dr \, d\theta \qquad u = 9 - r^2$$
$$du = -2r\,dr$$

$$= \frac{1}{2} \int_0^{2\pi} \int_5^9 \sqrt{u} \, du \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} u^{3/2} \Big|_5^9 \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (27 - 5\sqrt{5}) \, d\theta = \boxed{\frac{2\pi(27 - 5\sqrt{5})}{3}}$$

13.8.11 Calculate the volume of the solid bounded above by the sphere $r^2 + z^2 = 5$ and below by the paraboloid $r^2 = 4z$.



$$\int_0^{2\pi}\int_0^{2}\int_{\frac{r^2}{4}}^{\sqrt{5-r^2}} r\,dz\,dr\,d\theta$$

$$z^2 + 4z - 5 = 0$$

$$\frac{-4 + \sqrt{16 - 4(1)(-5)}}{2} = \frac{-4+6}{2} = 1$$

$$z = 1 \Rightarrow r = 2$$

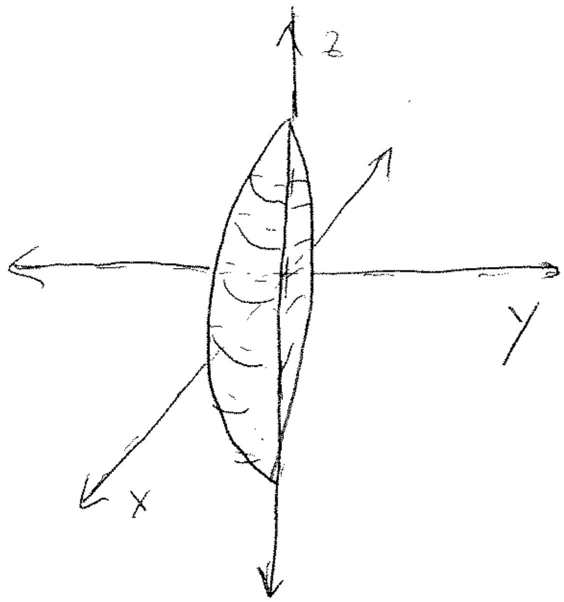
$$\int_0^{2\pi}\int_0^{2}\left(\sqrt{5-r^2}\,r - \frac{r^3}{4}\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{3}(5-r^2)^{3/2} - \frac{r^4}{16}\right)\Big|_0^2\,d\theta$$

$$= \int_0^{2\pi}\left[\left(-\frac{1}{3} - 1\right) - \left(-\frac{5\sqrt{5}}{3}\right)\right]d\theta = \int_0^{2\pi}\left(\frac{5\sqrt{5}-4}{3}\right)d\theta$$

$$= \boxed{\frac{2\pi(5\sqrt{5}-4)}{3}}$$

13.8.21 Calculate the volume of the smaller wedge cut from the unit sphere by two planes that meet at a diameter at an angle of 30°.



$$\int_0^{\pi}\int_0^{\pi/6}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\pi/6} \sin\phi \, d\theta \, d\phi$$

$$= \frac{\pi}{18}\int_0^{\pi} \sin\phi \, d\phi$$

$$= \frac{\pi}{18}(2) = \boxed{\frac{\pi}{9}}$$

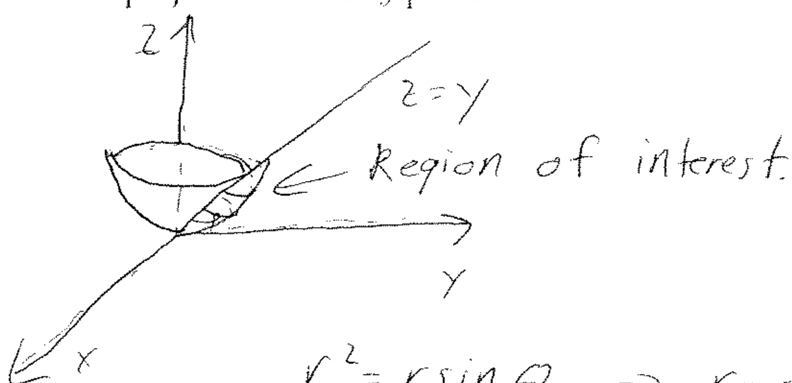
13.8.23 Find the volume of the solid bounded above by the plane:

$$z = y$$

and below by the paraboloid

$$z = x^2 + y^2.$$

Hint: In cylindrical coordinates the plane has equation $z = r\sin\theta$ and the paraboloid has equation $z = r^2$. Solve simultaneously to get the projection in the xy-plane.



$$r^2 = r\sin\theta \Rightarrow r = \sin\theta$$

$$\Rightarrow \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\sin\theta} (r^2\sin\theta - r^3)\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^3\sin\theta}{3} - \frac{r^4}{4}\Big|_0^{\sin\theta} d\theta \qquad \sin^4\theta = \left(\frac{1-\cos(2\theta)}{2}\right)^2$$

$$= \int_0^{\pi} \frac{\sin^4\theta}{12}\,d\theta = \frac{1}{12}\int_0^{\pi}\left(\frac{1-2\cos(2\theta)+\cos^2(2\theta)}{4}\right)d\theta$$

$$= \int_0^{\pi}\frac{1}{12}\left(\frac{1}{4} - \frac{\cos(2\theta)}{2} + \frac{1}{8}{}^{5} + \frac{\cos(4\theta)}{8}\right)d\theta \qquad \cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$$

$$= \frac{1}{12}\left(\frac{3\theta}{8} - \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32}\Big|_0^{\pi}\right) = \boxed{\cancel{\frac{3\pi}{8}}} = \boxed{\frac{\pi}{32}}$$