Math 2210 - Assignment 10

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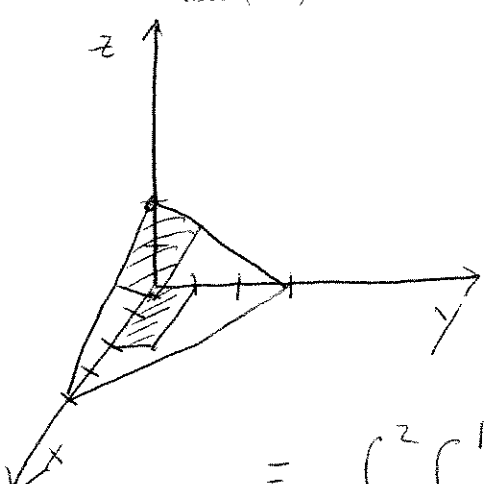
Sections 13.6 through 13.7

1 Section 13.6

13.6.1 Find the surface area of the part of the plane $3x + 4y + 6z = 12$ that is above the rectangle in the xy-plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$. Make a sketch of the surface.



$$z = 2 - \frac{1}{2}x - \frac{2}{3}y$$

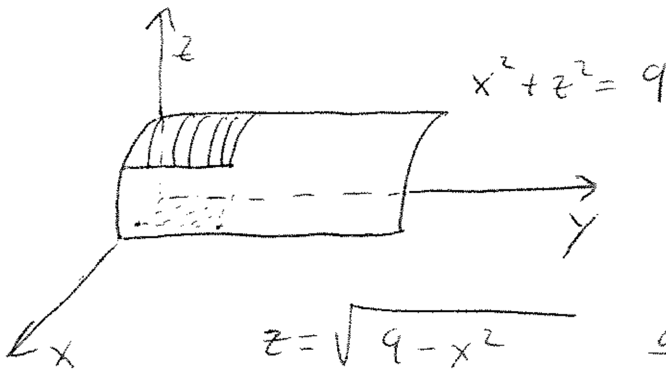
$$\frac{\partial z}{\partial x} = -\frac{1}{2} \quad \frac{\partial z}{\partial y} = -\frac{2}{3}$$

$$SA = \int_0^2 \int_0^1 \sqrt{1 + \left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{2}{3}\right)^2}\, dy\, dx$$

$$= \int_0^2 \int_0^1 \sqrt{\frac{36}{36} + \frac{9}{36} + \frac{16}{36}}\, dy\, dx$$

$$= \int_0^2 \int_0^1 \frac{\sqrt{61}}{6}\, dy\, dx = \int_0^2 \frac{\sqrt{61}}{6}\, dx = \boxed{\frac{\sqrt{61}}{3}}$$

13.6.5 Find the surface area of the part of the cylinder $x^2 + z^2 = 9$ that is directly over the rectangle in the xy-plane with vertices $(0,0)$, $(2,0)$, $(2,3)$, and $(0,3)$. Make a sketch of the surface.



$$z = \sqrt{9-x^2} \qquad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{9-x^2}} \qquad \frac{\partial z}{\partial y} = 0$$

$$SA = \int_0^2 \int_0^3 \sqrt{1 + \left(-\frac{x}{\sqrt{9-x^2}}\right)^2}\, dy\, dx$$

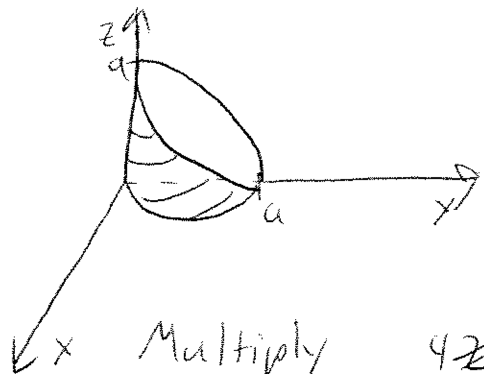
$$= \int_0^2 \int_0^3 \sqrt{\frac{9}{9-x^2}}\, dy\, dx$$

$$= 3 \int_0^2 \int_0^3 \frac{1}{\sqrt{9-x^2}}\, dy\, dx = 9 \int_0^2 \frac{dx}{\sqrt{9-x^2}}$$

$$= 9 \left[\sin^{-1}\left(\frac{x}{3}\right)\right] \Big|_0^2$$

$$= 9\left(\sin^{-1}\left(\frac{2}{3}\right) - \sin(0)\right) = \boxed{9 \sin^{-1}\left(\frac{2}{3}\right)}$$

13.6.12 Find the surface area of the part of the cylinder $x^2+y^2=ay$ inside the sphere $x^2+y^2+z^2=a^2$, $a>0$. *Hint*: Project to the yz-plane to get the region of integration. Make a sketch of the surface.



The surface area will be given by:

$$x=\sqrt{ay-y^2} \qquad \frac{\partial x}{\partial y}=\frac{a-2y}{2\sqrt{y(a-y)}} \qquad \frac{\partial x}{\partial z}=0$$

Multiply by ~~2~~4 to take in all ~~both~~ sides

$$4\cancel{2}\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{1+\left(\frac{a-2y}{2\sqrt{y(a-y)}}\right)^2}\,dx\,dy$$

$$=4\cancel{2}\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{\frac{4ay-4y^2+a^2-4ay+4y^2}{4(y(a-y))}}\,dx\,dy$$

$$=\cancel{4}2a\int_0^a\int_0^{\sqrt{a^2-y^2}}\frac{1}{\sqrt{y(a-y)}}\,dx\,dy = a\int_0^a\frac{\sqrt{a^2-y^2}}{\sqrt{y(a-y)}}\,dy$$

$$=2a\int_0^a\frac{\sqrt{a+y}}{\sqrt{y}}\,dy$$

~~$u=a\tan^2 y$~~ $y=a\tan^2 u$

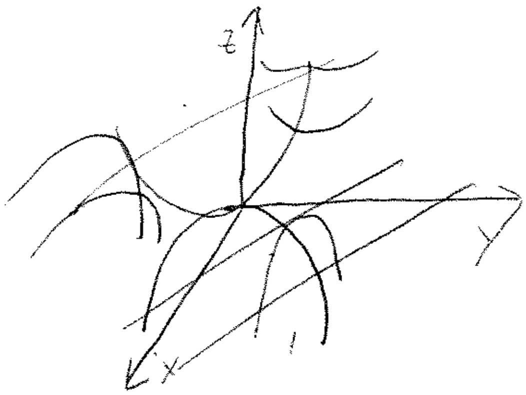
$dy=2a\tan u\sec^2 u$

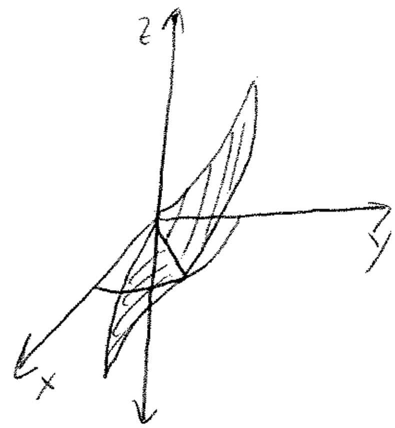
$$=\cancel{a\int_0^{\pi/4}\csc^3 u\,du}$$

$$=4a^2\int_0^{\pi/4}\sec^3 u\,du = 4a^2\left[\frac{1}{2}\sec u\tan u+\frac{1}{2}\ln|\sec u+\tan u|\right]\Big|_0^{\pi/4}$$

$$=\boxed{2a^2\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)}$$

13.6.13 Find the surface area of the part of the saddle $az = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$, $a > 0$. Make a sketch of the surface.





$$z = \frac{x^2 - y^2}{a} \qquad \frac{\partial z}{\partial x} = \frac{2x}{a} \qquad \frac{\partial z}{\partial y} = \frac{-2y}{a}$$

$$SA = \iint_S \sqrt{1 + \left(\frac{2x}{a}\right)^2 + \left(-\frac{2y}{a}\right)^2}\, dA = \frac{1}{a}\iint_S \sqrt{a^2 + 4x^2 + 4y^2}\, dA$$

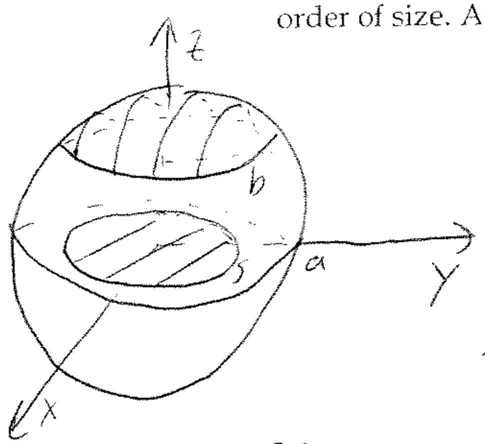
Converting to polar:

$$\frac{1}{a}\int_0^{2\pi}\int_0^a \sqrt{a^2 + 4r^2}\, r\,dr\,d\theta \qquad u = a^2 + 4r^2 \quad du = 8r\,dr$$

$$= \frac{1}{8a}\int_0^{2\pi}\int_{a^2}^{5a^2} \sqrt{u}\, du\, d\theta = \frac{1}{12a}\int_0^{2\pi} u^{3/2}\Big|_{a^2}^{5a^2} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12a}\left(5\sqrt{5} - 1\right) a^3\, d\theta = \boxed{\left(\frac{5\sqrt{5} - 1}{6}\right)\pi a^2}$$

13.6.21 Four goats have grazing areas A, B, C and D, respectively. The first three goats are each tethered by ropes of length b, the first on a flat plane, the second on the outside of a sphere of radius a, and the third on the inside of a sphere of radius a. The fourth goat must stay inside a ring of radius b that has been dropped over a sphere of radius a. Determine formulas for A, B, C and D and arrange them in order of size. Assume that $b < a$.



$$z = \sqrt{a^2 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$SA = \iint_S \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}}\, dA$$

$$= \iint_S \frac{a}{\sqrt{a^2-x^2-y^2}}\, dA \quad \text{converting to polar} \quad = a\int_0^{2\pi}\int_0^{b} \frac{r}{\sqrt{a^2-r^2}}\, dr\, d\theta$$

$u = a^2 - r^2$

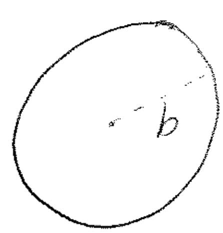
$-\frac{du}{2} = r\,dr$

$$= -\frac{a}{2}\int_0^{2\pi}\int_{a^2}^{a^2-b^2} \frac{du}{\sqrt{u}}\, du\, d\theta = \frac{a}{2}\int_0^{2\pi}\int_{a^2-b^2}^{a^2} \frac{du}{\sqrt{u}}\, du\, d\theta$$

$$= a\int_0^{2\pi}\left(a - \sqrt{a^2-b^2}\right) d\theta = \boxed{2\pi a\left(a - \sqrt{a^2-b^2}\right)}$$

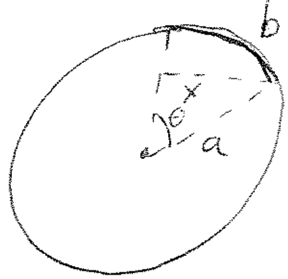
Situation A:

Grazing Area



$A = \pi b^2$

Situation B



$$\frac{b}{2\pi a} = \frac{\theta}{2\pi} \Rightarrow \theta = \frac{b}{a} \qquad \sin(\theta) = \frac{x}{a}$$

$$\Rightarrow SA = 2\pi a\left(a - \sqrt{a^2 - a^2\sin^2\theta}\right)$$

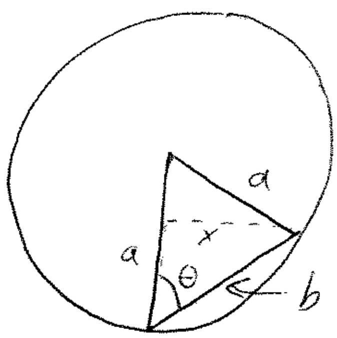
$$= 2\pi a^2(1 - \cos\theta) \qquad \cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$$

$$= \boxed{2\pi a^2\left(1 - \cos\left(\frac{b}{a}\right)\right)}$$

$$\cos\left(\frac{b}{a}\right) = 1 - \frac{b^2}{2a^2} + \cdots \qquad \text{So, } \approx \pi b^2 - \underbrace{\cdots}_{\text{smaller terms.}}$$

So, $B < A$

Situation C



Using the law of cosines

$$a^2 = a^2 + b^2 - 2ab\cos\theta \Rightarrow \cos\theta = \frac{b}{2a}$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{b}{2a}\right)^2 \Rightarrow \sin\theta = \sqrt{1 - \left(\frac{b}{2a}\right)^2}$$

$$\frac{x}{b} = \sin\theta = \sqrt{1 - \left(\frac{b}{2a}\right)^2} \Rightarrow x = b\sqrt{1 - \left(\frac{b}{2a}\right)^2}$$

So, the surface area is:

$$2\pi a\left(a - \sqrt{a^2 - \left(b\sqrt{1 - \frac{b}{2a}}\right)^2}\right) = 2\pi a\left(a - \sqrt{a^2 - b^2 + \frac{b^4}{4a^2}}\right)$$

$$= 2\pi a\left(a - \sqrt{\left(a - \frac{b}{2a}\right)^2}\right) = 2\pi a\left(a - \left(a - \frac{b^2}{2a}\right)\right) = 2\pi a\left(\frac{b^2}{2a}\right) = \boxed{\pi b^2}$$

Situation D

This is what we dealt with at the start, so:

$$SA = 2\pi a\left(a - \sqrt{a^2 - b^2}\right)$$

$$= \frac{2\pi a b^2}{a + \sqrt{a^2 - b^2}} > \pi b^2$$

So,

$A = \pi b^2$

$B = 2\pi a^2\left(1 - \cos\left(\frac{b}{a}\right)\right)$

$C = \pi b^2$

$D = 2\pi a\left(a - \sqrt{a^2 - b^2}\right)$

and $\boxed{B < A = C < D}$

2 Section 13.7

13.7.1 Evaluate the iterated integral:

$$\int_{-3}^{7}\int_{0}^{2x}\int_{y}^{x-1} dzdydx$$

$$\int_{-3}^{7}\int_{0}^{2x}\int_{y}^{x-1} dz\,dy\,dx$$

$$= \int_{-3}^{7}\int_{0}^{2x} z\Big|_{y}^{x-1} dy\,dx = \int_{-3}^{7}\int_{0}^{2x} (x-1-y)\,dy\,dx$$

$$= \int_{-3}^{7} xy - y - \frac{y^2}{2}\Big|_{0}^{2x} dx = \int_{-3}^{7} (2x^2 - 2x - 2x^2)\,dx$$

$$= -2\int_{-3}^{7} x\,dx = -x^2\Big|_{-3}^{7} = -49-(-9) = \boxed{-40}$$

13.7.5 Evaluate the iterated integral:

$$\int_4^{24}\int_0^{24-x}\int_0^{24-x-y}\frac{y+z}{x}\,dz\,dy\,dx$$

$$\int_4^{24}\int_0^{24-x}\int_0^{24-x-y}\frac{y+z}{x}\,dz\,dy\,dx = \int_4^{24}\int_0^{24-x}\frac{yz+\frac{z^2}{2}}{x}\,dy\,dx\,\Big|_0^{24-x-y}$$

$$= \int_4^{24}\int_0^{24-x}\frac{y(24-x-y)+\frac{(24-x-y)^2}{2}}{x}\,dy\,dx$$

$$= \int_4^{24}\int_0^{24-x}\frac{24y-xy-y^2+288-12x-12y-12x+\frac{x^2}{2}+\frac{xy}{2}-12y+\frac{xy}{2}+\frac{y^2}{2}}{x}\,dy\,dx$$

$$= \int_4^{24}\int_0^{24-x}\frac{-\frac{y^2}{2}+288-24x+\frac{x^2}{2}}{x}\,dy\,dx$$

$$= \int_4^{24} -\frac{y^3}{6x}+\frac{288y}{x}-24y+\frac{xy}{2}\,\Big|_0^{24-x}\,dx$$

$$= \int_4^{24}\left(-\frac{(24-x)^3}{6x}+\frac{288(24-x)}{x}-24(24-x)+\frac{x(24-x)}{2}\right)dx \qquad \cancel{u=24-x} \quad \cancel{du=-dx}$$

$$\cancel{=\int_0^{20}-\frac{u^3}{6(2\ldots}} = \int_4^{24}-\frac{2304}{x}+\cancel{96}\,288-12x+\frac{x^2}{6}-576+24x+12x-\frac{x^2}{2}$$

$$= \int_4^{24}\left(-\frac{2304}{x}-288+24x-\frac{x^2}{3}\right)dx = -2304\ln(x)-288x+12x^2-\frac{x^3}{9}\,\Big|_4^{24}$$

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$$= (-2304\ln(24)-6912+6912-1536)-\left(-2304\ln(4)-1152+192-\frac{64}{9}\right)$$

$$= -576-2304\ln(6)-\frac{64}{9}$$

Narsty

13.7.10 Evaluate the iterated integral:

$$\int_0^{\frac{\pi}{2}} \int_{\sin 2z}^{0} \int_0^{2yz} \sin\left(\frac{x}{y}\right) dx\,dy\,dz$$

$$\int_0^{\pi/2} \int_{\sin(2z)}^{0} \int_0^{2yz} \sin\left(\frac{x}{y}\right) dx\,dy\,dz$$

$$= \int_0^{\pi/2} \int_{\sin(2z)}^{0} -y\cos\left(\frac{x}{y}\right) \Big|_0^{2yz} dy\,dz$$

$$= \int_0^{\pi/2} \int_{\sin(2z)}^{0} (-y\cos(2z) + y)\,dy\,dz$$

$$= \int_0^{\pi/2} \frac{y^2}{2}(1-\cos(2z)) \Big|_{\sin(2z)}^{0} dz$$

$$= \int_0^{\pi/2} \frac{\sin^2(2z)}{2}(\cos(2z)-1)\,dz$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2(2z)\cos(2z)\,dz - \frac{1}{2}\int_0^{\pi/2} \sin^2(2z)\,dz$$

$u = \sin(2z)$

$\frac{du}{2} = \cos(2z)\,dz$

$$= \frac{1}{4}\int_0^{0} u^2\,du - \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos(4z)}{2}\,dz$$

$$= -\frac{1}{4}\left[z - \frac{\sin(4z)}{4}\right] \Big|_0^{\pi/2}$$

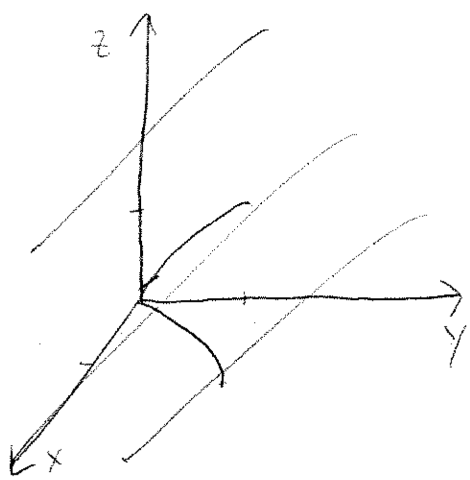
$$= -\frac{1}{4}\left[\frac{\pi}{2} - 0\right] - \left[-\frac{1}{4}(0-0)\right] = \boxed{-\frac{\pi}{8}}$$

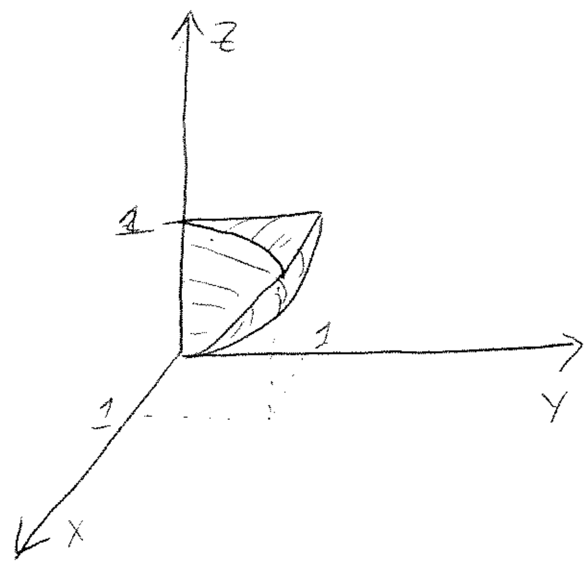
13.7.16 Sketch the solid:

$$S = \{(x, y, z) : 0 \le x \le y^2, 0 \le y \le \sqrt{z}, 0 \le z \le 1\}$$

and then write an iterated integral for:

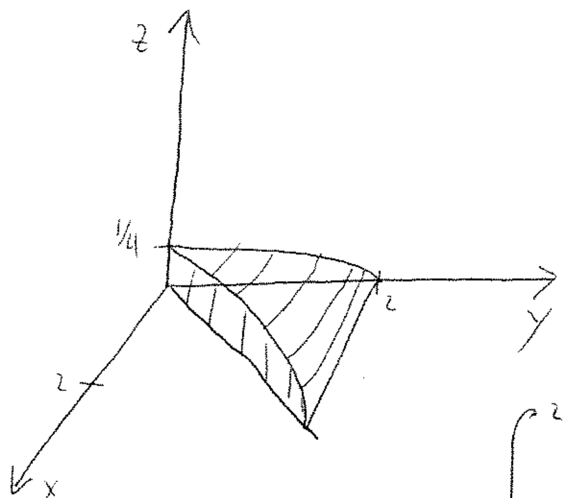
$$\int\int\int_S f(x, y, z)\,dV$$





$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

13.7.22 Calculate the volume of the solid in the first octant bounded by the elliptic cylinder $y^2 + 64z^2 = 4$ and the plane $y = x$.



$$\int_0^2 \int_0^y \int_0^{\frac{\sqrt{4-y^2}}{8}} dz\,dx\,dy$$

$$= \int_0^2 \int_0^y \frac{\sqrt{4-y^2}}{8}\,dx\,dy$$

$$= \int_0^2 \frac{y\sqrt{4-y^2}}{8}\,dy \qquad u = 4-y^2 \quad du = -2y\,dy$$

$$= \int_0^4 \frac{\sqrt{u}}{16} = \frac{u^{3/2}}{24}\Bigg|_0^4 = \frac{8}{24} = \boxed{\frac{1}{3}}$$