Midterm 2 - Finances, Exponentials, and Modeling

Math 1030 - Dylan Zwick's Class

Fall 2007

Name: Solutions

Financial Math Formulas

Compound Interest Formula for Annually Compounding Interest

$$A = P \times (1 + APR)^Y$$

Compound Interest Formula for Interest Paid n Times Per Year

$$A = P\left(1 + \frac{APR}{n}\right)^{(nY)}$$

Compound Interest Formula for Continuous Compounding

$$A = P \times e^{APR \times Y}$$

A = accumulated balance after Y years
P = starting principal
APR = annual percentage rate
Y = number of years

Savings Plan Formula (Regular Payments)
Note - Assumes same times for payments and compounding periods, which will be the case on all problems in this midterm.

$$A = PMT \times \frac{\left[\left(1 + \frac{APR}{n}\right)^{(nY)} - 1\right]}{\left(\frac{APR}{n}\right)}$$

A = accumulated savings plan balance
PMT = regular payment (deposit) amount
APR = annual percentage rate (as a decimal)
n = number of payment periods per year
Y = number of years

Exponential Formulas

Formula for Exponential Growth with the Doubling Time

$$y(t) = y_0 \times 2^{t/T_{double}}$$

Formula for Exponential Decay with the Doubling Time

$$y(t) = y_0 \times \left(\frac{1}{2}\right)^{t/T_{\frac{1}{2}}}$$

$y(t)$ = amount after time t
y_0 = initial amount (amount at t=0)
T_{double} = doubling time
$T_{\frac{1}{2}}$ = half-life

Logarithm Rules

$$\log(ab) = \log(a) + \log(b)$$
$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$
$$\log(a^b) = b\log(a)$$
$$\log_a(b) = \frac{\log(a)}{\log(b)}$$

Note that whenever a base is not specified that means it's true for any base, as long as the base is consistent throughout the equation.

1. **Compound and Simple Interest** (10 points)

Suppose you invest $1000 in an account today, planning to use it for a year long world trip when you retire in 40 years. If you invest at a 5% APR interest, compounding annually, how much money will you have for your trip when you retire? (5 points)

$$A = P \times (1 + APR)^{Y}$$
$$= \$1000 \times (1.05)^{40}$$
$$= \boxed{\$7,039.99}$$

How much money would you have if, instead of compounding annually, your interest compounded continuously? (4 points)

$$A = P \times e^{(APR \times Y)}$$
$$= \$1000 \times e^{(.05 \times 40)}$$
$$= \boxed{\$7,389.06}$$

What is the APY for continuous compounding when the APR is 5%. (1 point)

$$APY = e^{APR} - 1 = \boxed{5.127\%}$$

2. **Savings Plans** (10 points)

Suppose you're smart and start saving for retirement when you're 20 years old. You invest $\$1,000$ each month for 45 years. If your APR is 4.5%, compounded monthly, how much money will you have when you retire? ~~(5 points)~~ (4 points)

$$A = PMT \times \frac{\left[\left(1+\frac{APR}{n}\right)^{(nY)} - 1\right]}{\left(\frac{APR}{n}\right)}$$

$$= \$1,000 \times \frac{\left[\left(1+\frac{.045}{12}\right)^{(12\times 45)} - 1\right]}{\left(\frac{.045}{12}\right)}$$

$$= \cancel{\$1,341,150.68} \quad \boxed{\$1,745,992.37}$$

Suppose on the other hand that you would like to retire with $\$2,000,000$. Given the same set up as above (45 years, 4.5% APR, compounding monthly) what would your monthly payments need to be? (3 points)

$$PMT = \frac{A\left(\frac{APR}{n}\right)}{\left(1+\frac{APR}{n}\right)^{(nY)} - 1}$$

$$= \frac{\$2,000,000\left(\frac{.045}{12}\right)}{\left(1+\frac{.045}{12}\right)^{(12\times 45)} - 1}$$

$$= \boxed{\$1,145.48 \text{ /month}}$$

Suppose that instead of making payments every month you put down one lump sum at the start. What would this lump sum need to be for you to have $\$2,000,000$ when you retire, using the same setup as the first two parts of this problem. (3 points)

$$A = P \times \left(1+\frac{APR}{n}\right)^{(nY)}$$

$$P = \frac{\$2,000,000}{\left(1+\frac{.045}{12}\right)^{(12\times 45)}} = \boxed{\$264,989.41}$$

3. **Exponential Growth** (10 points)

State whether each is a case of linear or exponential growth: (3 points)

(a) A town whose population increases by 450 people every year.

Linear

(b) A town whose population increases by 3% every year.

Exponential

(c) Compound interest.

Exponential

If a town's population is 10,000 in 2007 and doubles every 5 years, what will its population be in 2022? (2 points)

$2022 - 2007 = 15$ years

$10,000 \times 2^{(15/5)} = \boxed{80,000}$

Suppose one bacteria starts out in a bottle at 9:00 AM and the number of bacteria in the bottle double every minute through mitosis. If the bottle is full at 10:00 AM, at what time will the bottle be half full? (2 points)

$\boxed{9:59\ AM}$

If you have an account that generates 5% APR interest and is compounded annually, how long does it take for invested money to double as a result of interest? In other words, if you put \$1000 in the account and left it there, how long would it take for you to earn enough interest so that you've got \$2000 in the account? (3 points)

$$T_2 = \frac{\log_{10} 2}{\log_{10}(1+.05)} = \frac{\log_{10}(2)}{\log_{10}(1.05)} = \boxed{14.21 \text{ years}}$$

4. **Functions and Logarithms** (10 points)

Can a function have more than one input with the same output? (Yes/No) (1 point)

Yes

Can a function have more than one output with the same input? (Yes/No) (1 point)

No

The domain of a function is all possible inputs? (True/False) (1 point)

True

The range of a function is all possible outputs? (True/False) (1 point)

True

For the function $f(x) = x^2$ is the function one-to-one? (Yes/No) (1 point)

No

What are the domain and range of the function $f(x) = x^2$? (2 points)

Domain: All real numbers

Range: All non-negative numbers

If an odd function is periodic with period 3 and if $f(1) = 2$ what is $f(-4)$? (3 points)

$f(1) = 2$ as period is 3

$f(4) = 2$ as function is odd

$\boxed{f(-4) = -2}$

5. **Linear and Exponential Models**

Suppose you start trick-or-treating with 10 candies, and each hour you get another 30 candies. How many candies do you have after 3 hours? (2 points)

$y(3) = 30 \times 3 + 10 = \boxed{100 \text{ candies}}$

What is the equation for the line that passes through the points $(3, 5)$ and $(5, 9)$? (5 points)

$m = \frac{\text{rise}}{\text{run}} = \frac{9-5}{5-3} = \frac{4}{2} = 2$

$y = 2x + b$ when $x = 3$ $y = 5$. So,

$5 = 2(3) + b \Rightarrow 5 = 6 + b \Rightarrow b = -1$

So, $\boxed{y = 2x - 1}$

If there are 100 mG of an antibiotic in the bloodstream, and the drug decreases exponentially with a half-life of 12 hours, how much antibiotic will be in the bloodstream after 15 hours? (3 points)

✓ concentration

$y(t) = y_0 \times \left(\frac{1}{2}\right)^{t/T_{1/2}}$

$y(15) = 100\,mg \left(\frac{1}{2}\right)^{(15/12)}$

$= \boxed{42\,mg}$

6. **Extra Credit**

Logarithms

Solve for x in the following equations:

(a) $47 = 3^x$ (2 points)

$\Rightarrow \log_{10} 47 = \log_{10}(3^x) = x \log_{10}(3)$

$\Rightarrow x = \frac{\log_{10}(47)}{\log_{10}(3)} = \boxed{3.50}$

(b) $x = \log_3(7)$ (2 points)

$x = \log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)} = \boxed{1.77}$

(c) $99 = 12 \times 4^x$ (2 points)

$\left(\frac{99}{12}\right) = 4^x \Rightarrow x = \frac{\log_{10}\left(\frac{99}{12}\right)}{\log_{10}(4)} = \boxed{1.52}$

For each of the following organizations or individuals indicate whether they are self declared Sunni or Shia Islam:

(3 points for 6/6, 2 points for 5/6, 1 point for 4/6)

(a) Hezbollah - Shia
(b) Nouri al-Maliki - Shia
(c) Hamas - Sunni
(d) Mahmoud Ahmadinejad - Shia
(e) Bashar al-Assad - Sunni
(f) Pervez Musharraf - Sunni

Who is the President of Iraq? (1 point)

Jalal Talibani