

Pascal Matrices and Particular Solutions to Differential Equations

John M. Zobitz*

March 21, 2003

Abstract

In this paper any non-homogeneous differential equation with constant coefficients is reduced to a matrix equation $\vec{q} = \vec{c}P$. For the discussion, \vec{q} represents a matrix of constant coefficients to the differential equation, \vec{c} a matrix of arbitrary constants to the solution, and P is a lower triangular matrix with entries that are derivatives of the characteristic polynomial of the differential equation. After careful development, the task becomes finding an inverse to the matrix P . Interestingly enough, P is a generalized form of what is termed a Pascal Matrix. [1] An inverse for certain conditions to such a matrix is proven to exist by the theorem given in the paper.

This approach was developed in earlier research, [2]. The advantage is that it uses fundamental concepts such as the linearity of the derivative, matrix multiplication, and product rule for derivatives. Furthermore a precise algorithm to solve a wide variety of differential equations is given with this approach.

1 Demonstration of Method.

How would one find a particular solution to the following differential equation?

$$y''' - y' + 3y = (1 + 5t)e^{4t} \quad (1)$$

We begin by defining an operator L so that $L = D^3 - D + 3$. In this instance D^k is the k^{th} derivative of y with respect to t . Note that:

$$L(e^{4t}) = 63e^{4t} = p(4)e^{4t} \text{ where } p(a) = a^3 - a + 3.$$

Let us assume a particular solution of $y^* = (c_0 + c_1t)e^{4t}$. Our strategy will be to differentiate the particular solution and compare it to the right hand side of Equation 1.

We can also rewrite the differential equation in matrix format:

*St. John's University

$$L(y) = \begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} e^{4t}.$$

By applying L to y^* we obtain the following:

$$L(y^*) = L\left(\begin{bmatrix} c_0 & c_1 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} e^{4t}\right) = \begin{bmatrix} c_0 & c_1 \end{bmatrix} L\left(\begin{bmatrix} 1 \\ t \end{bmatrix} e^{4t}\right).$$

By the linearity of L , one can apply L to each entry in the column vector. On the other hand, by direct calculation, $L(te^{4t}) = (63t + 47)e^{4t}$. These results can be subsequently written in matrix format:

$$\begin{bmatrix} c_0 & c_1 \end{bmatrix} \begin{bmatrix} L(e^{4t}) \\ L(te^{4t}) \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \end{bmatrix} \begin{bmatrix} 63 & 0 \\ 47 & 63 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} e^{4t} = \begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} e^{4t}.$$

As a result, this problem easily reduces to a simple matrix equation:

$$\begin{bmatrix} c_0 & c_1 \end{bmatrix} \begin{bmatrix} 63 & 0 \\ 47 & 63 \end{bmatrix} = \begin{bmatrix} 1 & 5 \end{bmatrix}.$$

The inverse of the 2×2 matrix is easily calculated to find c_0 and c_1 . So the particular solution to the differential equation is:

$$y^* = \left(\frac{-172}{3969} + \frac{5}{63}t\right) e^{4t}.$$

Certain questions naturally arise from this example.

1. How was the form of the particular solution determined?
2. If k and a are any real numbers, are there shortcuts to apply L to $t^k e^{at}$?
3. Is there an easy way other than the usual techniques to construct the 2×2 matrix that we inverted?
4. Is that matrix always invertible? What is its inverse?

Such questions lie at the heart of the method developed in this paper. These questions and others will be answered if we consider the problem more generally.

2 Theory.

2.1 Transforming the Problem.

Consider a n^{th} order differential equation with constant coefficients of the form:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = (q_0 + q_1 t + \cdots + q_m t^m) e^{at}. \quad (2)$$

where $n, m \geq 0$ and a_k and q_k represent constant coefficients.

Define a linear operator $L(y)$ such that:

$$L(y) = \sum_{k=0}^n a_k D^k(y),$$

where D^k is the k^{th} derivative with respect to t . Thus the left hand side of Equation 2 is represented by $L(y)$. Furthermore, the right hand side of Equation 2 can be represented in matrix form:

$$(q_0 + q_1 t + \cdots + q_m t^m) e^{at} = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots & q_m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}.$$

So Equation 2 is transformed into the following equation:

$$L(y) = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots & q_m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}. \quad (3)$$

2.2 Finding a particular solution.

We assume a particular solution y^* of the form $y^* = (c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m) e^{at}$. We do this by selecting the order of the polynomial $c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m$ to be of the same order as the polynomial on the right hand side of Equation 3. The plan is to compute $L(y^*)$ and then compare it to the right hand side of Equation 3. This will allow us to determine the coefficients c_i .

Writing y^* in matrix format we can see that:

$$\begin{aligned} L(y_p) &= L \left(\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at} \right) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix} L \left(\begin{bmatrix} e^{at} \\ t e^{at} \\ t^2 e^{at} \\ \vdots \\ t^m e^{at} \end{bmatrix} \right) \\ &= \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_m \end{bmatrix} \begin{bmatrix} L(e^{at}) \\ L(t e^{at}) \\ L(t^2 e^{at}) \\ \vdots \\ L(t^m e^{at}) \end{bmatrix}. \end{aligned} \quad (4)$$

Now we develop a formula for $L(t^k e^{at})$ where $0 \leq k \leq m$. Observe that:

$$L(e^{at}) = \sum_{k=0}^n a_k D^k(e^{at}) = \sum_{k=0}^n a_k a^k e^{at} = e^{at} \sum_{k=0}^n a_k a^k = p(a)e^{at},$$

where $p(a) = \sum_{k=0}^n a_k a^k$.

Note that $p(a)$ is the characteristic polynomial to the differential equation. To find higher derivatives of $L(e^{at})$ we note two things. First, by considering $L(e^{at})$ as a function of a and t , we can utilize the fact that in this case the order of differentiation of mixed partial derivatives can be interchanged. Using this we see:

$$L(t^k e^{at}) = L\left(\frac{\partial^k(e^{at})}{\partial a^k}\right) = \frac{\partial^k(L(e^{at}))}{\partial a^k} = \frac{\partial^k(p(a)e^{at})}{\partial a^k}.$$

Second, from Reference [3] we can invoke Leibinz's Rule for higher derivatives of the product of two functions u and v :

$$(uv)^{(n)} = uv^{(n)} + \binom{n}{1} u'v^{(n-1)} + \dots + vu^{(n)} = \sum_{r=0}^n \binom{n}{r} u^{(r)}v^{(n-r)}.$$

Thus, to calculate $L(t^k e^{at})$ we apply this rule to $\frac{\partial^k(p(a)e^{at})}{\partial a^k}$ and simplify:

$$\begin{aligned} L(t^k e^{at}) &= \frac{\partial^k(p(a)e^{at})}{\partial a^k} = \sum_{l=0}^k \binom{k}{l} [p(a)]^{(k-l)} [e^{at}]^{(l)} \\ &= \sum_{l=0}^k \binom{k}{l} p^{(k-l)} t^l e^{at} = e^{at} \sum_{l=0}^k \binom{k}{l} p^{(k-l)} t^l, \end{aligned} \quad (5)$$

where $p^{(k-l)}$ signifies the $k-l$ th derivative of p with respect to a .

Note that $\sum_{l=0}^k \binom{k}{l} p^{(k-l)} t^l$ is just a polynomial with coefficients involving derivatives of p . By writing out the terms, we can use a $m+1 \times m+1$ matrix P to represent Equation 5.

Let P_{k+1} represent the $k+1$ th row of P . Naturally we would like to organize P so that $L(e^{at}) = L(t^0 e^{at}) = p(a)e^{at}$ corresponds to entry P_{11} of the matrix. Since k ranges from 0 to m , $L(t^k e^{at})$ will correspond to row $k+1$. Similarly, we order the columns by the power of t in Equation (5) which is l .

Note then that P is a lower triangular matrix, so $P_{kl} = 0$ for $k < l$. Thus,

$$P_{k+1,l+1} = \begin{cases} \binom{k}{l} p^{(k-l)} & k \geq l \\ 0 & k < l \end{cases} \text{ where } 0 \leq k, l \leq m \quad (6)$$

By writing the powers of t as a column vector we have the following:

$$L \left(\begin{bmatrix} e^{at} \\ te^{at} \\ t^2 e^{at} \\ \vdots \\ t^m e^{at} \end{bmatrix} \right) = \begin{bmatrix} p & 0 & 0 & \cdots & 0 \\ p' & p & 0 & \cdots & 0 \\ p'' & 2p' & p & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p^{(m)} & \binom{m}{1} p^{(m-1)} & \cdots & \binom{m}{m-1} p' & p \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at} \quad (7)$$

Set $\vec{c} = [c_0 \ c_1 \ c_2 \ \cdots \ c_m]$ and $\vec{q} = [q_0 \ q_1 \ q_2 \ \cdots \ q_m]$.
Comparing Equations 3 and 7 we obtain:

$$L(y_p) = \vec{c}P \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at} = \vec{q} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}. \quad (8)$$

Thus Equation 8 reduces to the matrix equation:

$$\vec{q} = \vec{c}P. \quad (9)$$

P is a general form of a Pascal Matrix, a lower triangular matrix with entries that correspond to Pascal's Triangle. Reference [1] only discusses Pascal Matrices with integer entries. In this situation the entries of P are higher order derivatives of a function. With Reference [1] as a guide, we can generalize an inverse to such matrices to find an inverse for P , which is what we need to solve the differential equation.

2.3 Finding a Solution.

In order to solve for \vec{c} , we need to find P^{-1} . In order for P^{-1} to exist, $\det(P)$ must be nonzero. Since P is a lower triangular matrix, $\det(P) = [p(a)]^m$. For the moment, assume that $p(a) \neq 0$. (The case $p(a) = 0$ will be considered later.)

Theorem 1 *Let P be as in Equation 6 and assume $p(a) \neq 0$. Let*

$$Q_{k+1,l+1} = \begin{cases} \binom{k}{l} \left(\frac{1}{p}\right)^{(k-l)} & k \geq l \\ 0 & k < l, \end{cases} \quad (10)$$

then $P^{-1} = Q$.

PROOF: It is clear from properties of lower triangular matrices that:

$$(PQ)_{k+1,l+1} = \begin{cases} 0 & k < l \\ 1 & k = l. \end{cases}$$

Because when $k < l$ the matrix entry will be zero in any case. If $k = l$, then $k - l = 0$ and $p^{(k-l)} = p$. Similarly, $(1/p)^{(k-l)} = 1/p$, and the product of the two is 1.

Now we need to show that $(PQ)_{k+1,l+1} = 0$ if $k > l$. Suppose $k > l$. Then $k = l + q$ for some $q > 0$. We see that:

$$\begin{aligned} (PQ)_{k+1,l+1} &= \sum_{r=0}^q P_{l+q+1,l+r+1} Q_{l+r+1,l+1} \\ &= \sum_{r=0}^q \binom{l+q}{l+r} \binom{l+q}{l} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}. \end{aligned} \quad (11)$$

Expanding the binomial terms, Equation 11 is equivalent to:

$$(PQ)_{k+1,l+1} = \sum_{r=0}^q \frac{(l+q)!(l+r)!}{(l+r)!(q-r)!r!} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}.$$

If we multiply both the numerator and denominator by $q!$ and factor out the terms not dependent on r we obtain:

$$(PQ)_{k+1,l+1} = \frac{(l+q)!}{l!q!} \sum_{r=0}^q \frac{q!}{(q-r)!r!} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}. \quad (12)$$

or:

$$(PQ)_{k+1,l+1} = \binom{l+q}{q} \sum_{r=0}^q \binom{q}{r} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}.$$

Since $k = l + q$ we have,

$$(PQ)_{k+1,l+1} = \binom{k}{q} \sum_{r=0}^q \binom{q}{r} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)}. \quad (13)$$

Using Leibniz's Rule for Higher Derivatives of Products once more, Equation 13 becomes:

$$(PQ)_{k+1,l+1} = \binom{k}{q} \sum_{r=0}^q \binom{q}{r} p^{(q-r)} \left(\frac{1}{p}\right)^{(r)} = \binom{k}{q} \left(\frac{1}{p} \cdot p\right)^{(q)}.$$

Since $q > 0$, $\binom{k}{q} \left(\frac{1}{p} \cdot p\right)^{(q)} = \binom{k}{q} (1)^{(q)} = 0$. Thus, when $k > l$, $(PQ)_{k+1,l+1} = 0$.

So, $(PQ)_{k+1,l+1} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$ Showing that $P^{-1} = Q$.

For our purposes p is specified as the characteristic polynomial of the differential equation, yet this theorem generalizes the results of [2] to include functions, instead of integers, as entries in the Pascal Matrix.

Now that P^{-1} is found, Equation 9 can be solved and the coefficients of the vector \vec{c} can be determined. Thus, the particular solution y^* to the differential equation is obtained, that is:

$$\vec{c} = \vec{q}P^{-1} \text{ and } y^* = \vec{c}\vec{t}e^{at}. \quad (14)$$

A good exercise would be to use the methods described and apply them to the initial example. As one can see, we used the same method outlined thus far to solve the first example.

2.4 Adjusting the Method.

The restriction that $p(a) \neq 0$ is a limitation. Note that finding P^{-1} hinged on the assumption that P has a nonzero determinant. Consider the following differential equation:

$$y'' - 16y = (1 + t^2)e^{4t}. \quad (15)$$

Applying our method to this differential equation reveals that $p(4) = 0$. This gives a matrix P with 0's along the main diagonal, which makes its determinant zero. Thus the technique seems to fail.

If $p^{(j)}(a) = 0$ for all $0 \leq j < q < m$, this suggests that y_h , the homogenous solution to Equation 2 is:

$$\sum_{j=0}^{q-1} r_j t^j e^{at} \text{ where } r_j \text{ is a constant.}$$

A lower triangular matrix can still be constructed to determine the particular solution. In this case we assume a particular solution of the form:

$$(c_q t^q + c_{q+1} t^{q+1} + \dots + c_m t^m + \dots + c_{q+m} t^{q+m})e^{at}.$$

To find the value of each c_i we can follow a similar process outlined previously. Since $p^{(j)}(a) = 0$ for $j < q$, the corresponding entries in Equation 7 will be zero. As a result, Equation 7 reduces to a $m + 1 \times m + 1$ lower triangular matrix:

$$L \left(\begin{bmatrix} t^q e^{at} \\ t^{q+1} e^{at} \\ t^{q+2} e^{at} \\ \vdots \\ t^{q+m} e^{at} \end{bmatrix} \right) = \begin{bmatrix} p^{(q)} & 0 & 0 & \dots & 0 \\ p^{(q+1)} & \binom{q+1}{1} p^{(q)} & 0 & \dots & 0 \\ p^{(q+2)} & \binom{q+2}{1} p^{(q+1)} & \binom{q+2}{2} p^{(q)} & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p^{(q+m)} & \binom{q+m}{1} p^{(q+m-1)} & \dots & \binom{q+m}{m-1} p^{(q+1)} & \binom{q+m}{m} p^{(q)} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} e^{at}.$$

Thus, we can extend our previous formulation of matrix P to the following:

$$P_{k+1,l+1} = \begin{cases} \binom{q+k}{l} (p^{(q)})^{(k-l)} & k \geq l \\ 0 & k < l \end{cases} \quad (16)$$

where $0 \leq k, l \leq m$ and $p^{(j)} = 0$ for $j < q$. Since $p^{(q)} \neq 0$, $\det(P) \neq 0$ and P^{-1} exists.

Returning to Equation 15, let's assume a particular solution of the form:

$$y^* = (c_1 t + c_2 t^2 + c_3 t^3) e^{4t}$$

Since $p'(4) = 8$, by emphasizing the binomial coefficients, we see that this leads to a matrix P :

$$\begin{bmatrix} 1 \cdot 8 & 0 & 0 \\ 1 \cdot 2 & 2 \cdot 8 & 0 \\ 1 \cdot 0 & 3 \cdot 2 & 3 \cdot 8 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 16 & 0 \\ 0 & 6 & 24 \end{bmatrix}.$$

One may innocently think that the form of P^{-1} would be similar to (10). Unfortunately, this is not correct. The key to finding an P^{-1} rested in the fact that P had the form of a Pascal Matrix. Equation 16 certainly has entries from Pascal's Triangle yet that is not a strong enough condition to guarantee that in this case P^{-1} can be generalized. As a result, we need to find P^{-1} from standard techniques.

For completion, P^{-1} for Equation 15 is given by:

$$\begin{bmatrix} \frac{1}{8} & 0 & 0 \\ -\frac{1}{64} & \frac{1}{16} & 0 \\ \frac{1}{256} & -\frac{1}{64} & \frac{1}{24} \end{bmatrix}.$$

So the particular solution to Equation 15 is:

$$y^* = \left(\frac{33}{256} t - \frac{1}{64} t^2 + \frac{1}{24} t^3 \right) e^{4t}.$$

2.5 Comments.

- If the order n of the differential equation is less than the order m of the row vector \vec{q} , then $p^{(j)}(a) = 0$ for $n \leq j \leq m$. Due to the restriction that $p(a) \neq 0$, it will never be the case that P will be a zero matrix. In fact, if $m = 0$, the particular solution is quite simple:

$$y^* = \frac{q_0}{p(a)e^{at}} \text{ when } m = 0.$$

- As Gollwitzer remarks, this method can be used to solve a wide variety of non-homogeneous equations. If faced with a trigonometric equation on the right-hand side, one can use Euler's identity and set $a = i\omega$. The final solution will either be the real or the imaginary part of the particular solution. Table 1 summarizes these adjustments to Equation 14:

Right hand side of Equation 2	Adjustment to Equation 14
$q_0 + q_1 t + \cdots + q_m t^m$	$a = 0$
$(q_0 + q_1 t + \cdots + q_m t^m) \sin(\omega t)$	$a = i\omega$, Im(Equation 14).
$(q_0 + q_1 t + \cdots + q_m t^m) \cos(\omega t)$	$a = i\omega$ Re(Equation 14).

Table 1: Adjustments to Equation 14.

3 Conclusions.

Non-homogeneous differential equations with constant coefficients arise frequently in physics, chemistry, and engineering. For example, forced motion of a pendulum and LRC circuits generate such differential equations. In practice, this method could be applied to many cases encountered by a physicist, chemist, biologist, as well as a mathematician.

The following procedure can be applied to solve most differential equations of the form given in Equation 2:

1. Identify $p(a)$, the characteristic polynomial and m , the number that determines the size of P .
2. Construct $Q = P^{-1}$ from (10).
3. Multiply \vec{q} and P^{-1} to find \vec{c} , the coefficients to the particular solution.
4. Make necessary adjustments to Table 1 if needed.

As it can be seen, this procedure could be implemented by a computer program. In general, computers could solve Equation 2 with less time using this method than with a method such as undetermined coefficients because this method utilizes matrices, which generally take less computational time.

4 Acknowledgements.

- Many thanks to my advisor, Dr. Shobha Deshmukh, for bringing this project to my attention and to her gentle guidance and patience while working through it.
- Thanks to Dr. Jennifer Galovich for her careful revisions of this work. Her editing skills made this more readable.

References

- [1] Call G.S and Vellman D.J., *Pascal's Matrices*, American Mathematical Monthly, April 1993, pp 372-376.

- [2] Gollwitzer H., *Matrix Patterns and Undetermined Coefficients*, College Mathematics Journal, Vol. 25, No. 5, pp 445-448, 1994.
- [3] Speigel M.R. and Liu John, "Mathematical Handbook of Formulas and Tables," 2nd ed, McGraw Hill, 1999.

5 About the author

John M. Zobitz
Department of Mathematics
University of Utah
155 S. 1400 E. Rm 233
Salt Lake City, UT 84112-0090
email: zobitz@math.utah.edu

John Zobitz is a first year graduate student at the University of Utah studying mathematical biology. His interests are the outdoors and biking. In May 2002 he graduated from St. John's University with a B.A. in Mathematics and Spanish.