Pattern Formation in Arid Ecosystems

A Bifurcation Analysis

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The Plan


Vegetation Patterns

- Examples of Patterns
- Mechanisms for Patterns
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Examples of Patterns

The left picture shows examples of bushy patterns in (b) Niger, (c) Israel, (e) French Guiana, and (f)-(g) are peatlands in western Siberia from Rietkerk et al 2004.
Mechanisms for Patterns

Some mechanisms for this patterning include:

- Positive feedbacks between vegetation and water:
  - Soil water is locally redistributed from depth by deep roots.
  - This redistribution allows plants to grow locally.
  - Increased shading (reduced evaporation) means more water is around.
Mechanisms for Patterns (cont.)

- Short-range facilitation/long-range competition for resources
  - Long-superficial roots compete for sparse nutrients
  - Positive local feedback for growth.

Vegetation patterns and uniform vegetation due to local facilitation, long-range competition in Ivory Coast from Rietkerk et al 2004
Mechanisms for Patterns (cont.)

• “Ecosystem engineers” (cyanobacteria, plants, microorganisms)
  • Associate with shrubs to locally accumulate soil-water.

→ Environmental changes cause catastrophic species loss!
As a reminder...
Modeling Vegetation Patterns

Non-dimensionalized, the model presented in Meron et al 2004 is the following:

\[
\frac{\partial n}{\partial t} = \frac{\gamma w}{1 + \sigma w} n - n^2 - \mu n + \Delta n \\
\frac{\partial w}{\partial t} = p - (1 - \rho n)w - w^2 n + \delta \Delta (w - \beta n)
\]

Where:

- \( n \): Plant density
- \( w \): Amount of water
Where each of the terms are:

- \( \gamma w \frac{n}{1 + \sigma w} \): Plant growth dependent on water availability
- \( n^2 \): Herbivory
- \( \mu n \): Mortality
- \( \Delta n \): Dispersal
- \( p \): Incoming precipitation
- \(- (1 - \rho n) w \): Evaporation reduced by plant shading
- \( w^2 n \): Water loss due to transpiration
- \( \delta \Delta (w - \beta n) \): Soil water transport via Darcy’s Law.

Two key parameters will be \( p \) and \( \rho \)
Bifurcation Analysis

Without considering space, there is an equilibrium at:

\[ n = 0, \quad w = p \]

This solution can be continued via the Implicit Function Theorem until:

\[ p_c = \frac{\mu}{\gamma - \mu \sigma} \]
Bifurcation Analysis (cont.)

Shift our system by $\tilde{w} = w - p$, then at equilibrium, we must satisfy:

$$\dot{n} = A(\rho)n^3 + B(p, \rho)n^2 + C(p, \rho)n + D(p, \rho) = 0$$

Varying $\rho$ will change the cubic structure. A fold bifurcation will arise when we project the curve $\Gamma$ onto the plane:

$$\Gamma = \begin{cases} 
A(\rho)n^3 + B(p, \rho)n^2 + C(p, \rho)n + D(p, \rho) = 0 \\
3A(\rho)n^2 + 2B(p, \rho)n + C(p, \rho) = 0
\end{cases}$$
Bifurcation Analysis (cont.)

The places where the stability changes are where \( \frac{\partial p}{\partial n} = 0 \), or solutions of:

\[
\Gamma = \begin{cases} 
A(\rho)n^3 + B(p_c, \rho)n^2 + C(p_c, \rho)n = 0 \\
\frac{\partial p}{\partial n} = 0
\end{cases}
\]

We should obtain two values \( p_0 \) and \( p_1 \).
A critical value $\rho_c$ can be found by solving the following system:

$$\Gamma_c = \begin{cases} A(\rho)n^3 + B(p_c, \rho)n^2 + C(p_c, \rho)n = 0 \\ \frac{\partial p}{\partial n}\bigg|_{p=p_c} = 0 \end{cases}$$

Values greater than $\rho_c$ will induce a threshold value $p_0 < p_c$. 

![Diagram of bifurcation analysis](image-url)
Assume we have appropriate eigenfunctions of the Laplacian. Linearize about an equilibrium point $n_0, w_0$, with the following Jacobian:

$$
J = \begin{bmatrix}
\frac{\gamma w_0}{1+\sigma w_0} - 2n_0 - \mu & \frac{\gamma n_0}{(1+\sigma w_0)^2} \\
\frac{w_0(\rho - w_0)}{(1+\sigma w_0)^2} & \frac{n_0(\rho - 2w_0) - 1}{(1+\sigma w_0)^2}
\end{bmatrix}
$$

$$
n = \sum_k c_k e^{\lambda t} N_k(r) \quad (3)
$$

$$
\tilde{w} = \sum_k \tilde{c}_k e^{\lambda t} W_k(r) \quad (4)
$$
We obtain the following eigenvalues:

\[
\lambda_{\pm} = \frac{1}{2} \left( -b(k^2) \pm \sqrt{b(k^2)^2 - 4h(k^2)} \right),
\]

(5)

where:

\[
b(k^2) = k^2(1 + \delta) + n_0 + \frac{p}{w_0} + w_0 p_0
\]

(6)

\[
h(k^2) = \det J + \delta k^4 - k^2(J_{22} + \delta J_{11} - \delta \beta J_{12})
\]

(7)
\( \lambda_+ < 0 \) when \( h(k^2) \geq 0 \). The uniform state undergoes a fold bifurcation and becomes unstable for finite wavenumbers \( k \) when:

\[
\det J = \frac{(J_{22} + \delta J_{11} - \delta \beta J_{12})^2}{4\delta}
\]

Solving this for \( p \) gives critical points \( (p_2) \) for which precipitation values stabilize or destabilize the uniform vegetation state, leading to non-uniform patterns.

We can also define a critical wavenumber:

\[
k_c = \sqrt{\frac{\det J}{\delta}}
\]
Putting it all together

We can classify new types of *desertification*:

- $p > p_2$: *(dry-subhumid)* Uniform vegetation is stable
- $p_1 < p < p_2$: *(semiarid)* The only possible states are non-uniform vegetation patterns
- $p_0 < p < p_1$: *(arid)* All three stable states are possible.
- $p < p_0$: *(hyperarid)* Only the bare state is stable.
For more information...