Pattern Formation in Arid Ecosystems

A Bifurcation Analysis

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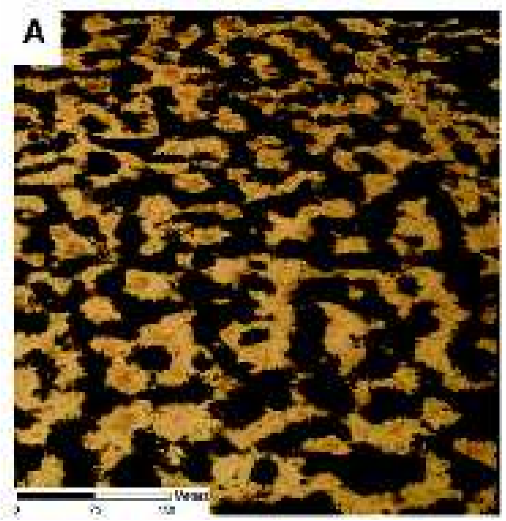
The Plan

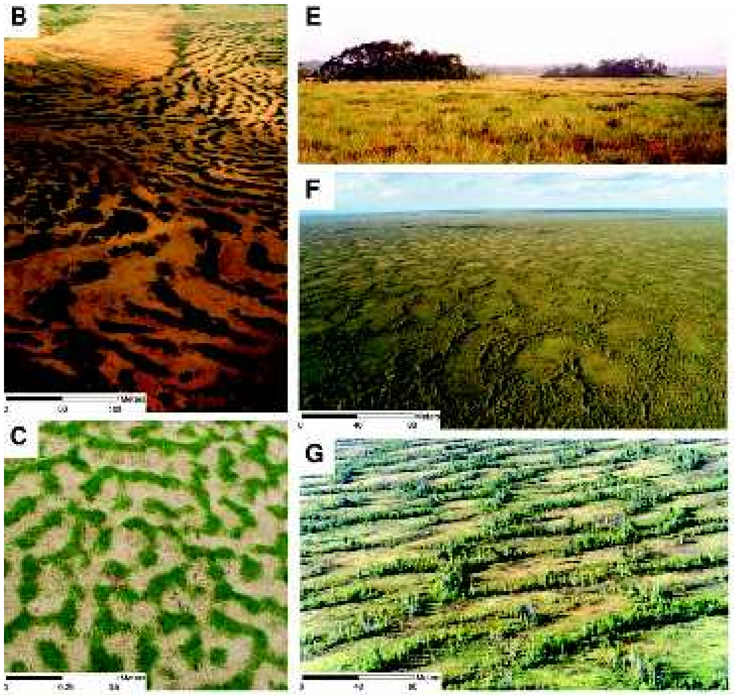


Bushy vegetation in Niger, from Rietkerk et al 2004.

Vegetation Patterns

- Examples of Patterns
- Mechanisms for Patterns
- Modeling Vegetation Patterns
- Bifurcation Analysis
- Spatial Aspects

Examples of Patterns



The left picture shows examples of bushy patterns in (b) Niger, (c) Israel, (e) French Guiana, and (f)-(g) are peatlands in western Siberia from Rietkerk et al 2004

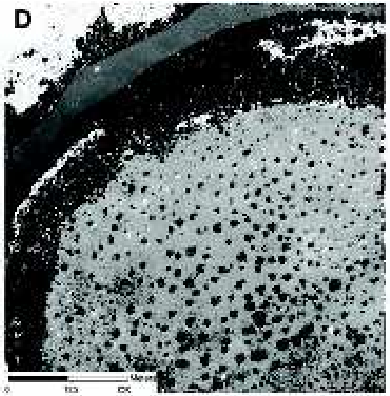
Mechanisms for Patterns

Some mechanisms for this patterning include:

- Positive feedbacks between vegetation and water:
  - Soil water is locally redistributed from depth by deep roots.
  - This redistribution allows plants to grow locally.
  - Increased shading (reduced evaporation) means more water is around.

Mechanisms for Patterns (cont.)

- Short-range facilitation/long-range competition for resources
  - Long-superficial roots compete for sparse nutrients
  - Positive local feedback for growth.



Vegetation patterns and uniform vegetation due to local facilitation, long-range competition in Ivory Coast from Rietkerk et al 2004

Mechanisms for Patterns (cont.)

- “Ecosystem engineers” (cyanobacteria, plants, microorganisms)
  - Associate with shrubs to locally accumulate soil-water.

$\rightarrow$ Environmental changes cause catastrophic species loss!

As a reminder...

Modeling Vegetation Patterns

Non-dimensionalized, the model presented in Meron et al 2004 is the following:

$$\frac{\partial n}{\partial t} = \frac{\gamma w}{1+\sigma w} n - n^2 - \mu n + \Delta n \qquad (1)$$

$$\frac{\partial w}{\partial t} = p - (1 - \rho n) w - w^2 n + \delta \Delta (w - \beta n) \qquad (2)$$

Where:

- n: Plant density
- w: Amount of water

Modeling Vegetation Patterns (cont.)

Where each of the terms are:

- $\frac{\gamma w}{1+\sigma w} n$: Plant growth dependent on water availability
- n^2: Herbivory
- μn: Mortality
- Δn: Dispersal
- p: Incoming precipitation
- $-(1 - \rho n)w$: Evaporation reduced by plant shading
- $w^2 n$: Water loss due to transpiration
- $\delta \Delta (w - \beta n)$: Soil water transport via Darcy's Law.

Two key parameters will be p and ρ

Bifurcation Analysis

Without considering space, there is an equilibrium at:

$$n = 0, \quad w = p$$

This solution can be continued via the Implicit Function Theorem until:

$$p_c = \frac{\mu}{\gamma - \mu\sigma}$$

Bifurcation Analysis (cont.)

Shift our system by $\tilde{w} = w - p$, then at equilbrium, we must satisfy:

$$\dot{n} = A(\rho)n^3 + B(p,\rho)n^2 + C(p,\rho)n + D(p,\rho) = 0$$

Varying ρ will change the cubic structure. A fold bifurcation will arise when we project the curve Γ onto the plane:

$$\Gamma = \begin{cases} A(\rho)n^3 + B(p,\rho)n^2 + C(p,\rho)n + D(p,\rho) = 0 \\ 3A(\rho)n^2 + 2B(p,\rho)n + C(p,\rho) = 0 \end{cases}$$

Bifurcation Analysis (cont.)

The places where the stability changes are where $\partial p/\partial n = 0$, or solutions of:

$$\Gamma = \begin{cases} A(\rho)n^3 + B(p_c,\rho)n^2 + C(p_c,\rho)n = 0 \\ \frac{\partial p}{\partial n} = 0 \end{cases}$$

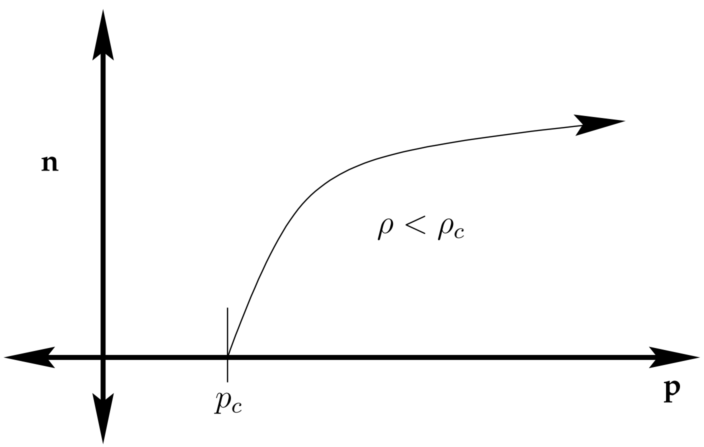
We should obtain two values p_0 and p_1.

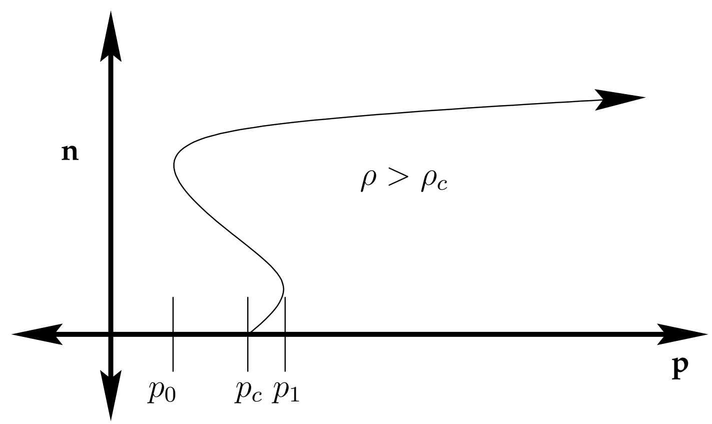
Bifurcation Analysis (cont.)

A critical value ρ_c can be found by solving the following system:

$$\Gamma_c = \begin{cases} A(\rho)n^3 + B(p_c,\rho)n^2 + C(p_c,\rho)n = 0 \\ \frac{\partial p}{\partial n}\Big|_{p=p_c} = 0 \end{cases}$$

Values greater than ρ_c will induce a threshold value $p_0 < p_c$.





Spatial Aspects

Assume we have appropriate eigenfunctions of the Laplacian. Linearize about an equilibrium point n_0, w_0, with the following Jacobian:

$$J = \begin{bmatrix} \frac{\gamma w_0}{1+\sigma w_0} - 2n_0 - \mu & \frac{\gamma n_0}{(1+\sigma w_0)^2} \\ w_0(\rho - w_0) & n_0(\rho - 2w_0) - 1 \end{bmatrix}$$

$$n = \sum_k c_k e^{\lambda t} N_k(r) \qquad (3)$$

$$w = \sum_k \tilde{c}_k e^{\lambda t} W_k(r) \qquad (4)$$

Spatial Aspects (cont.)

We obtain the following eigenvalues:

$$\lambda_{\pm} = \frac{1}{2}\left(-b(k^2) \pm \sqrt{b(k^2)^2 - 4h(k^2)}\right), \tag{5}$$

where:

$$b(k^2) = k^2(1+\delta) + n_0 + \frac{p}{w_0} + w_0 p_0 \tag{6}$$

$$h(k^2) = \det J + \delta k^4 - k^2(J_{22} + \delta J_{11} - \delta\beta J_{12}) \tag{7}$$

Spatial Aspects (cont.)

$\lambda_+ < 0$ when $h(k^2) \geq 0$. The uniform state undergoes a fold bifurcation and becomes unstable for finite wavenumbers k when:

$$\det J = \frac{(J_{22} + \delta J_{11} - \delta\beta J_{12})^2}{4\delta}$$

Solving this for p gives critical points (p_2) for which precipitation values stabilize or destabilize the uniform vegetation state, leading to non-uniform patterns.

We can also define a critical wavenumber:

$$k_c = \sqrt{\frac{\det J}{\delta}}$$

Putting it all together

We can classify new types of *desertrification*:

- $p > p_2$: *(dry-subhumid)* Uniform vegetation is stable
- $p_1 < p < p_2$: *(semiarid)* The only possible states are non-uniform vegetation patterns
- $p_0 < p < p_1$: *(arid)* All three stable states are possible.
- $p < p_0$: *(hyperarid)* Only the bare state is stable.

For more information...

- J. Von Hardenberg, E. Meron et al. Diversity of Vegetation Patterns and Desertrification. 2001. Physical Review Letters. 87(19). doi:10.1103/PhysRevLett.87.198101
- E. Meron, E. Gilad, et al. Vegetation Patterns along gradient. 2004. Chaos, Solitons, and Fractals. 19:367-376.
- M. Rietkerk, S. C. Dekker. Self-Organized Patchiness and Catastrophic Shifts in Ecosystems. 2004. Science. 305:1926-1929.
- M. R. Aguiar and Osvaldo E. Sala. Patch structure, dynamics and implications for the functioning of arid ecosystems. 1999. TREE. 14(7):273-277.