

Key

Name: _____

Quiz II

Instructions: This quiz is a total of 30 points with each question worth 15 points. You are to answer ONE of Problem 1 and 2, and you MUST answer Problem 3. Answer each question carefully and thoughtfully to receive full credit. Partial credit will be awarded, and points will be deducted if you write the answer down to a problem without justifying your steps. You do not need to simplify your answer unless it helps for clarity. Calculators are not permitted (nor are needed) for this quiz.

1. Find a basis for the space of all lower triangular 3 by 3 matrices. What is the dimension of this space?

Consider $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ to be a typical element of $L^{3 \times 3}$

this is written as

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so span of all these vectors form a basis
dimension = 6

2. Answer the following:

- Define what it means for a linear transformation to be an isomorphism.
- Is the transformation $T(x + iy) = x$ from \mathbb{C} to \mathbb{C} a linear transformation? (What do you need to show or to find?)
- Is the transformation in Part b an isomorphism? Why or why not?

a. A linear transformation is an isomorphism if its invertible or if $\ker(T) = \{0\}$ and $\text{im}(T) = W$

b. $T(x+iy)$ is linear (verify $T(x+iy + u+iv) = x+u = T(x+iy) + T(u+iv)$
 $T(k(x+iy)) = kx = kT(x+iy)$)

or the matrix of this transformation is

$$T(x+iy) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

c. No \Rightarrow the kernel is nontrivial. (All functions iy (real part 0))
map to 0

3. Consider the following transformation T from the space of upper triangular matrices ($U^{2 \times 2}$ to $U^{2 \times 2}$):

$$T(M) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} M - M \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

with respect to the following basis \mathcal{B} :

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

a. Is this transformation T an isomorphism?

b. If T is not an isomorphism, then what are the bases of $\text{im}(T)$ and $\text{ker}(T)$?

The following may be of use:

$$T \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \begin{bmatrix} 0 & -2b \\ 0 & 0 \end{bmatrix}$$

a. No it is not $\rightarrow \text{ker}(T)$ includes the identity, so $\text{ker}(T) \neq \{0\}$

b. we know $T \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = \begin{bmatrix} 0 & -2b \\ 0 & 0 \end{bmatrix},$

so if $x = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and

$$\left[T \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) \right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2b \\ 0 \end{bmatrix},$$

so the B matrix is:

$$[T(x)]_{\mathcal{B}} = B[x]_{\mathcal{B}},$$

and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\text{im}(B) = \text{span} \left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right)$$

which in our system correspond to

$$\text{ker}(B) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\text{im}(T) = \text{span} \left(\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \right)$$

$$\text{ker}(T) = \text{span} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$