Practice Final Key

1. a. $\lambda = 1, 2, 4$

$$E_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad E_2 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad E_4 = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

so $\quad S^{-1} D S = A \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

b $\quad A^t = S^{-1} D^t S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

c. Since $\quad Ax = \lambda x \qquad A(Ax) = A(\lambda x)$

$$A^2 x = \lambda A x = \lambda^2 x \quad \ldots$$

so $\quad A^3$ has eigenvalues $1, 8, 64$

eigenvectors are the same.

d. If $\quad Ax = \lambda x \qquad (\lambda, x)$ is eigenpair

an eigen pair of A^t is (λ^t, x)

2. $tr(A) = 0 = \lambda_1 + \lambda_2 \Rightarrow \lambda_1 = -\lambda_2$

$\det A = 9 = \lambda_1 \lambda_2$ so $(-\lambda_2)(\lambda_2) = 9$

$\lambda_2^2 = -9 \qquad \lambda_2 = \pm 3i$

so eigenvalues are $\pm 3i$

3. 3 volume $= \sqrt{\det(A^T A)}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{pmatrix} = A^T A$$

$$\det A^T A = 1 \cdot \det \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 1 \\ 10 & 30 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

$$20 - 20 + 6 = 6$$

so volume $= \sqrt{6}$

4. $A = \begin{bmatrix} 4 & 3 \\ 4 & 8 \end{bmatrix} \qquad f_A(\lambda) = \lambda^2 - 12\lambda + 20 = 0$

$\rightarrow (\lambda - 10)(\lambda - 2)$

a. $\lambda = 10, 2$

b. Since $\lambda = 10, 2 > 1$, for the discrete dynamical system 0 is unstable

c. $\lambda_{1,2} > 0$, so 0 is unstable as well.

5. Eigenvalues of A: $0, 3, -3$

Eigenvectors:

$\lambda = 0$: $\vec{v}_1 = [1, -2, 2]^T \rightarrow \vec{u}_1 = \frac{1}{3}[1, -2, 2]^T$

$\lambda = 3$ $\vec{v}_2 = [2, 2, 1]^T$ $\quad \vec{u}_2 = \frac{1}{3}[2, 2, 1]^T$

$\lambda = -3$ $\vec{v}_3 = [-2, 1, 2]^T$ $\quad \vec{u}_3 = \frac{1}{3}[-2, 1, 2]^T$

$$\underbrace{\vec{u}_1, \vec{u}_2, \vec{u}_3}_{\text{orthonormal eigenbasis}}$$

6. Recall

$$\begin{array}{ccc} M & \xrightarrow{T} & T(M) \\ \downarrow & & \downarrow \\ [M]_B & \xrightarrow{A} & [T(M)]_B \end{array}$$

$B = \{M_1, M_2, M_3, M_4\}$

so

$$A = \begin{bmatrix} [T(M_1)]_B & [T(M_2)]_B & \cdots & [T(M_4)]_B \end{bmatrix}$$

For each element in the basis

so $T(M_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad T(M_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T(M_3) = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} = 3M_3$ $\quad T(M_4) = \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} = 3M_4$

so $\begin{bmatrix} [T(M_1)]_B & [T(M_2)]_B & [T(M_3)]_B & [T(M_4)]_B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = A$

a basis for

$$\ker(B) = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$$ (verify this)

b.

$$\text{im}(B) = \text{span}\left\{\begin{pmatrix}0\\0\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\3\end{pmatrix}\right\}$$

c. so a basis for the kernel of T is $\begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix}$

image of T. $\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}$

d. rank(T) = 2

nullity(T) = 2 $\quad$ <u>NOT</u> an isomorphism

7. The reduced row echelon form of A is:

$$\begin{bmatrix}1 & 2 & 0 & 3 & 0\\0 & 0 & 1 & -4 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$
$\uparrow v_1 \quad \uparrow v_2 \quad \uparrow v_3 \quad \uparrow v_4 \quad \uparrow v_5$

v_1 * $\quad v_3$ * $\quad v_5$ *

These have leading 1's

$v_2 = 2v_1 \quad \rightarrow \quad 2v_1 - v_2 = 0$

$v_4 = -4v_3 + 3v_1 \quad\quad 3v_1 - 4v_3 - v_4 = 0$

$$\ker(A) = \text{span}\left\{\begin{pmatrix}2\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}3\\0\\-4\\-1\\0\end{pmatrix}\right\}$$

$$\text{im}(A) = \text{span}\left\{\begin{pmatrix}1\\-1\\4\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\5\\1\end{pmatrix}, \begin{pmatrix}6\\1\\9\\-7\end{pmatrix}\right\}$$

8. Note that the columns of M are linearly independent so apply Gram-Schmidt to the columns.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$v_2^{\perp} = v_2 - (u_1 \cdot v_2)u_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - 1 \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$\|v_2^{\perp}\| = 1$, so $v_2^{\perp} = u_2$

$u_1 \cdot v_3 = 2\left(\frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2} = 2$

$u_2 \cdot v_3 = 2\left(-\frac{1}{2}\right) - \frac{1}{2} + \frac{1}{2} = 1$

$$v_3^{\perp} = v_3 - (u_1 \cdot v_3)u_1 - (u_2 \cdot v_3)u_2$$

$2 - 1 + \frac{1}{2}$

$$\begin{bmatrix}0\\2\\1\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}-3/2\\3/2\\1/2\\-1/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-3\\3\\1\\-1\end{bmatrix}$$

$$\|v_3^{\perp}\| = \frac{\sqrt{20}}{2} = \sqrt{5}$$

so $$Q = \begin{bmatrix}1 & 1 & -3\\1 & -1 & 3\\1 & -1 & 1\\1 & 1 & 1\end{bmatrix} \qquad R = \begin{bmatrix}\|v_1\| & (u_1 \cdot v_2) & (u_1 \cdot v_3)\\ & \|v_2^{\perp}\| & (u_2 \cdot v_3)\\ & & \|v_3^{\perp}\|\end{bmatrix}$$

$$= \begin{bmatrix}2 & 1 & 2\\0 & 1 & 1\\0 & 0 & \sqrt{5}\end{bmatrix}$$

9. We know that eigenvalues solve $\det(A - \lambda I) = 0$.

If A is not invertible, then $\det(A) = 0$

$\det(A) = \det(A - 0 \cdot I) = 0$, so 0 must be an eigenvalue of A

10. Notice that row II is a multiple of row 1, so we need a least squares solution.

$$A^T A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

$\det A^T A = 0$, but solve

$$\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 5 & 15 & 6 \\ 15 & 45 & 18 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 5 & 15 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

set $x_2 = t$, $5x_1 + 15x_2 = 6 \Rightarrow x_1 = \frac{6}{5} - \frac{15x_2}{5}$

$x_1 = \frac{6}{5} - 3x_2$

solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} - 3t \\ t \end{bmatrix}$