HW 5

5.1 9, 13, 16

5.2 4, 19, 32 → 36

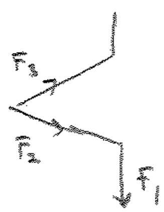
5.3 5-26 (even), 29, 33

5.4 1, 4, 10, 13, 25, 37

$$\begin{array}{r} 14 \\ 5 \\ \hline 90 \\ 36 \\ \hline 126 \end{array}$$

9. $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|} = \dfrac{3 \cdot 4 + 5 - 3 \cdot 3}{2 \cdot \sqrt{59}} = \dfrac{1}{2\sqrt{59}}$ ⇒ acute angle. (in 1^{st} quadrant)

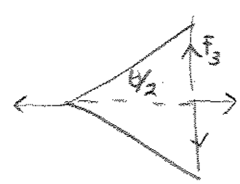
13.



$\|F_2\| = \|F_1\| = \|F_3\|$

The Forces $F_2 + F_3$ Have vertical and Horizontal components. The vertical components cancel.

$F_{leg} = 16.$



So $(F_3)_x$ (Force of 3 in x direction) is equal to

$F_3 \cos\left(\frac{\theta}{2}\right)$. Same as $(F_1)_x$

Adding these up.

$$F_{leg} = F_3 \cos\left(\frac{\theta}{2}\right) + F_1 \cos\left(\frac{\theta}{2}\right)$$

$$16 = 2(10)\cos\left(\frac{\theta}{2}\right) \Rightarrow .8 = \cos\left(\frac{\theta}{2}\right)$$

so $\theta = 2 \cdot \arccos(.8) = 74^\circ$

16. We want $\tilde{x}$ to be perpendicular to $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$.

Denote $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ The matrix formed by

We must satisfy

$\tilde{u}_1 \cdot x = 0$ → this is a matrix equation

$\tilde{u}_2 \cdot x = 0$

$u_3 \cdot x = 0$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

so $x_1 = x_4$

$x_2 = -x_4$

$\cancel{x_3 = -x_4} \Rightarrow x_4 = -x_3.$

and the solution set is spanned by $t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = v_1$

$\|v_1\| = \pm 2t$

but we want a unit vector, so

$$u_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

5.2

4. using Gram-Schmidt on $\begin{bmatrix}4\\0\\3\end{bmatrix}, \begin{bmatrix}25\\0\\-25\end{bmatrix}, \begin{bmatrix}0\\-2\\0\end{bmatrix}$ gives

$$\begin{bmatrix}4/5\\0\\3/5\end{bmatrix}, \begin{bmatrix}3/5\\0\\-4/5\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

19. QR (A), $A = \begin{bmatrix}2 & 1\\2 & 1\\1 & 5\end{bmatrix}$ is

$$Q = \begin{bmatrix}-2/3 & -1/\sqrt{18}\\ -2/3 & -1/\sqrt{18}\\ -1/3 & 4/\sqrt{18}\end{bmatrix} \qquad R = \begin{bmatrix}3 & 3\\ 0 & \sqrt{18}\end{bmatrix} \qquad \sqrt{18} = 4.24$$

32. Find an orthonormal basis of $x_1 + x_2 + x_3 = 0$

spanned by : $x_1 = -x_2 - x_3 \qquad x_2 = s \qquad x_3 = t$,

~~and~~ or $\text{span}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$

using Gram-Schmidt, an orthonormal basis of this is

$$\text{span}\left(\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}, \begin{bmatrix}-\sqrt{6}/6\\\sqrt{6}/6\\-\sqrt{6}/6\end{bmatrix}\right)$$

5.3 5-26 (even) (True means orthogonal)

5 T

6. T

8. $\|(A+B)x\| = \|Ax + Bx\| \neq \|Ax\| + \|Bx\|$ F

10. $\|B^{-1}ABx\| = \|B^{-1}(ABx)\| = \|A(Bx)\| = \|Bx\| = \|x\|$ T

12. ?

3 pts each

14. Symmetric

16. $(A+B)^T = A^T + B^T = A + B$ so symmetric

18. $(A^{10})^T = (A \cdots A)^T = A^T \cdots A^T = A^{10}$ so symmetric

20. $(AB^2A)^T = A^T(B^2)^T A^T = AB^2A$ symmetric

12×3=36

22. symmetric (BB^T)

24. not symmetric no

26 symmetric (addition is associative)

29. consider $\cos\theta = \dfrac{L(v)\cdot L(w)}{\|L(v)\|\,\|L(w)\|} = \dfrac{L(v)\cdot L(w)}{\|v\|\,\|w\|} = \dfrac{\vec{v}\cdot\vec{w}}{\|v\|\,\|w\|}$ (exercise 28)

so Let $L(v) = Av$. so $L(\vec{v})\cdot L(w) = (Av)\cdot(Aw)$
$= v^TA^TAw$
$= v^Tw = v\cdot w$

The converse, we have $\dfrac{v^TA^TAw}{\|Av\|\,\|Aw\|} = \dfrac{v^Tw}{\|v\|\,\|w\|}$

If pick $w = \begin{bmatrix}1\\0\end{bmatrix}$, $v = \begin{bmatrix}0\\1\end{bmatrix}$, so the right hand side is 0.

then let $A = \begin{bmatrix}a & 0\\ c & d\end{bmatrix}$, $A^TA = \begin{bmatrix}a^2+c^2 & cd\\ cd & d^2\end{bmatrix}$

$\begin{bmatrix}a & c\\ 0 & d\end{bmatrix}\begin{bmatrix}a^2+c^2 & cd\\ cd & d^2\end{bmatrix}$

and $A^TAw = \begin{bmatrix}a^2+c^2\\ cd\end{bmatrix}$ $v^TA^TAw = cd = 0.$ set $d=0,$

~~then we have~~ $A = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$, and this is certainly not orthogonal.

$$A^T A = \begin{bmatrix} a^2+c^2 & 0 \\ 0 & 0 \end{bmatrix} \neq I_2.$$

33 Find all orthogonal 2x2 matrices

set $v_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, for arbitrary θ. This is a unit vector.

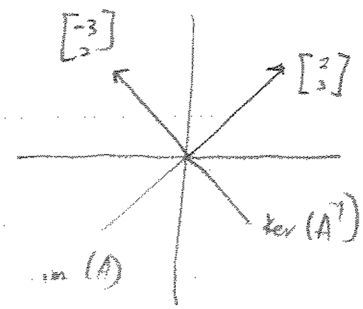
Any vector orthonormal to v_1 will either be $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, or $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$,

So the choices are $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

for arbitrary θ

5.4

1. A basis of $\ker(A^T)$ is $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$



4. We know $(\operatorname{im} B)^{\perp} = \ker(B^T)$

Let $B = A^T$, so $(\operatorname{im} A^T)^{\perp} = \ker(A)$

Take the orthogonal complement of both sides $(V^{\perp})^{\perp} = V$.

$(\operatorname{im} A^T) = (\ker A)$

10. a. If $\vec{x}$ is an arbitrary solution to the system $Ax=b$, let
$x_h = \operatorname{proj}_V(\vec{x})$, $V = \ker(A)$.

$x_0 = \vec{x} - \operatorname{proj}_V(\vec{x})$ $\quad b = Ax = A(x_h + x_0) = Ax_0$, $(x_h \in \ker(A))$

b. If x_0 & x_1 are two solutions of $Ax=b$, (From $(\ker A)^{\perp}$),
$x_1 - x_0$ is in $(\ker A)^{\perp}$ as well.

$A(x_1 - x_0) = Ax_1 - Ax_0 = b - b = 0$, so $x_1 - x_0 \in \ker(A)$
so $x_1 - x_0 \in \ker(A)$, and $x_1 - x_0 \in (\ker A)^{\perp}$, so $x_1 - x_0 = 0 \Rightarrow x_1 = x_0$

c. write $x_1 = x_h + x_0$ in part a. x_h is orthogonal to x_0,
so $\|x_0\| \leq \|x_1\|$ From Pythagorean theorem.

13. a. S'pose $L^+(y_1) = x_1$ $L^+(y_2) = x_2$. so x_1 & x_2 are both in $(\ker(A))^\perp = \mathrm{im}(A^T)$

so then $A^TAx_1 = A^Ty_1$, $A^TAx_2 = A^Ty_2$.
and $x_1 + x_2$ is in $\mathrm{im}(A^T)$ as well,

$$A^TA(x_1+x_2) = A^T(y_1+y_2), \text{ so } L^+(y_1+y_2) = x_1+x_2$$

verifying $L^+(ky) = kL^+(y)$ is analogous.

b. $L^+(L(x))$ is the orthogonal projection of x onto $(\ker A)^\perp = \mathrm{im}(A^T)$
This follows from problem 10, when you showed that the minimal solution lives in $(\ker(A))^\perp$ and Exercise 12 where you showed that the minimal least squares solution lives in $(\ker A)^\perp$.

c. $L(L^+(y))$ is the orthogonal projection of $\vec{y}$ onto $\mathrm{im}(A)$.

d. $\mathrm{im}(L^+) = \mathrm{im}(A^T)$ $\ker(L^+) = \ker(A^T)$ (parts b & c)

e. $L(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = Ax$

L^+ is the least squares solution of $L(x) = y \Rightarrow Ax = y$

$$x^* = \underbrace{(A^TA)^{-1}A^T}_{\text{this is } L^+} y$$

$$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4x_1$$

$A^Ty = [2y_1 \ 0 \ 0]^T$ so since $4x_1 = 2y_1 \Rightarrow x_1 = \frac{1}{2}y_1$

so $L^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

25. The normal equation
$A^TAx = A^Tb$

$$\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$ reduces to

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ so $x_1 + 3x_2 = 1$

and $x_1 = 1 - 3x_2$

solutions are of the form $x^* = \begin{bmatrix} 1-3t \\ t \end{bmatrix}$

37 We want c_0, c_1 where

$$\begin{aligned} c_0 + c_1(35) &= \log 35 \\ c_0 + c_1\, 46 &= \log 46 \\ c_0 + c_1\, 59 &= \log 77 \\ c_0 + c_1\, 69 &= \log 133 \end{aligned}$$

$$\begin{bmatrix} 1 & 35 \\ 1 & 46 \\ 1 & 59 \\ 1 & 69 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \log 35 \\ \log 46 \\ \log 77 \\ \log 133 \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* = (A^TA)^{-1}A^Tb \approx \begin{bmatrix} 0.915 \\ 0.017 \end{bmatrix}$$ $\log d \approx 0.915 + 0.017t$

b $d \approx 10^{0.915} \cdot 10^{.017t} \approx 8.22 \cdot 10^{.017t}$

c. If $t = 88$, $d \approx 258$. Since it has 93 displays, technologies have rendered it obsolete