Chapter 4

4.1 4, 17, 30, 45
4.2 13, 22, 48, 51, 54, 62
4.3 1, 7, 8, 36, 42, 47, 61.

4.1

4 Is $\{ p(t) \; \int_0^1 p(t)\,dt = 0 \}$ a subspace of P_2?

P_2: $ax^2 + bx + c$ it contains the 0 vector (a = b = c = 0)

Is $k p(t) + l q(t)$ in the set, where $\int_0^1 p(t)\,dt = 0$, $\int_0^1 q(t)\,dt = 0$?

yes: $\int_0^1 k\,p(t) + l\,q(t)\,dt = k\int_0^1 p(t)\,dt + l\int_0^1 q(t)\,dt = 0$ subspace

Basis: $\int_0^1 at^2 + bt + c\,dt = 0$

$\Rightarrow \frac{a}{3} + \frac{b}{2} + c = 0$, let: $b = s$, $c = t$

Basis = span$\left(-\frac{3}{2}t^2 + t,\; -3t^2 + 1\right)$

17 A basis of $\mathbb{R}^{m \times m}$ are matrices w/ 1 in a entry and 0 everywhere else its dimension is mn

30. The space of all matrices A such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$a + 2c = 0$
$b + 2d = 0$
$3a + 6c = 0$
$3b + 6d = 0$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 6 & 0 \\ 0 & 3 & 0 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find this kernel,

using maple, we find that ~~the solution~~ ,

$a = -2c$ so let $c = s$

$b = -2d,$ $d = t,$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2s \\ -2d \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

So all

$$A = \text{span}\left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \right)$$ and the basis is two dimensional.

45. Find a basis of all matrices that commute with

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, we want to find A such that $AB = BA$

or $AB - BA = 0$.

using maple, $AB = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$ $BA = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$

$AB - BA =$

$$\begin{bmatrix} -d & a-e & b-f \\ -g & d-h & e-i \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ we see automatically that

$d = g = h = 0$

So we have

$a-e=0$
$b-f=0$
$e-i=0$

3 equations, 5 unknowns

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ e \\ f \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the kernel of this matrix is

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

c is another free variable, (we've assigned $d=g=h=0$, and a,b,e,f,i

so A is of the form:

$$\operatorname{span}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

and this is of dimension 3

4.2

13 $T(M) = M\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}M$ from $\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$

is a linear transformation $\to$ we are multiplying matrices and subtracting are by definition linear transformation.

Let $M = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then

$$T(M) = \begin{bmatrix}-2c & 2a-2d\\0 & 2c\end{bmatrix}$$

Ker (T) is the matrix equation

$T(M) = 0$. ~~From above we see that~~ $c=0$,

$2a-2d = 0 \Rightarrow a = d$,

and b is unspecified,

so $\ker(T) = \operatorname{span}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)$

so it is not an isomorphism $\Rightarrow \dim(\ker(T)) = 2$

22. $T(f(t)) = \int_{-2}^{3} f(t)\,dt$ from $P_2 \to \mathbb{R}$

is certainly a linear transformation (integration is linear)

Let $f(t) = at^2 + bt + c$

so $T(f(t)) = \frac{35}{3}a + \frac{5b}{2} + 5c$ $\frac{7}{3}a + \frac{1}{2}b + c$

solving $T(f(t)) = 0$ gives 2 free variables, and

$$\ker(T) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{3}{14}s - \frac{3}{7}t \\ s \\ t \end{bmatrix}$$

$$\ker(T) = \text{span}\left(\begin{bmatrix} -3/14 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/7 \\ 0 \\ 1 \end{bmatrix} \right) \quad -\frac{3}{14}t^3 + t \quad , \quad -\frac{3}{7}t^2 + 1$$

48. $T(f(t)) = f'(t)$ from P to P.
This is a linear space. (derivatives can be differentiated and are linear)
but this is not isomorphic; because $\ker(T) \neq 0$.

Let $f = 0$, then $f' = 0$,
but if $f = 1$, then $f' = 0$, and

$1 \in \ker(T)$,
so $\ker(T)$ is spanned by $\{1\}$, NOT an isomorphism.

51. From Before (see problem 13)

$$\ker = \text{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

nullity is 2.

54. From prob 22.

$$\ker(T) = \text{span}\left(\begin{bmatrix} -3/14 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/7 \\ 0 \\ 1 \end{bmatrix} \right)$$

nullity is 2.

62. From #48,
ker = span(1).
nullity is 1.

4.3 1, 7, 8, 36, 42, 47, 61

If the polynomials $f(t) = 7 + 3t + t^2$
$g(t) = 9 + 9t + 4t^2$
$h(t) = 3 + 2t + t^2$ are linearly independent, then

$$A = \begin{bmatrix} 7 & 9 & 3 \\ 3 & 9 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$ must have full rank, or $\ker(A) = 0$.

$\mathrm{rref}(A) = I_3$, so yes they are

7. The matrix of the transformation is the coordinates of T applied to each basis vector
If the kernel of the matrix is 0, it is an isomorphism

the matrix of the transformation is: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

kernel spanned by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

image spanned by $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

8. The matrix of the transformation is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

kernel is spanned by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

image spanned by $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

36. $T(M) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} M - M \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$

if $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(M) = \begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}$

so the coordinates of M are

$$\begin{tikzcd} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \arrow[r, "T"] \arrow[d] & \begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix} \arrow[d] \\ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \arrow[r, "B"] & \begin{bmatrix} b-c \\ a-d \\ d-a \\ c-b \end{bmatrix} \end{tikzcd}$$

$$B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

the kernel of B is: $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$

and the image of B is: $\text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$

<u>Not</u> an isomorphism

in $\mathbb{R}^{2\times 2}$, the kernel is spanned by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and the image is spanned by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

42

a. The S matrix is $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

b. Verify $SB = AS$

c. the change of basis from $\mathcal{U} \to \mathcal{B}$ is given by S^{-1}.

$$S^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

47

$$\mathcal{U} = (1, t, t^2)$$
$$\mathcal{B} = (1, t-3, (t-3)^2)$$

Let $f(t) = a + bt + ct^2 = c_1 + c_2(t-1) + c_3(t-1)^2$

$$= c_1 + c_2 t - c_2 + c_3(t^2 - 2t + 1)$$

$$= (c_1 + c_2 + c_3) + (c_2 - 2c_3)t + c_3 t^2$$

so $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

↑

this is $S_{B \to \mathcal{U}}$

b. Verify $SB = AS$

c. $S_{M \to B} = (S_{B \to \mathcal{U}})^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

61. Consider $\mathcal{U} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right)$

$\mathcal{B} = \left(\begin{bmatrix} 5 \\ -10 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \end{bmatrix} \right)$

$S_{\mathcal{B}\to\mathcal{U}}$ is given by the coordinates of $\mathcal{B}$ with respect to $\mathcal{U}$

$$\begin{bmatrix} 5 \\ -10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

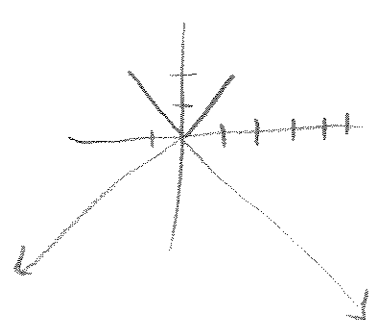
$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

and $\begin{bmatrix} 10 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

so $S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

b. $S_{\mathcal{U}\to\mathcal{B}} = \left(S_{\mathcal{B}\to\mathcal{U}} \right)^{-1} = \frac{1}{25} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

show $\begin{bmatrix} 5 & 10 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$



to its a reflection and scaling