95

2 $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}_{R_2 - 2R_1} \rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}_{\frac{1}{2}R_1} \rightarrow \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$

$x_1 = -\frac{3}{2}x_2$ so the vectrs are $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
$x_2 = s$

20 we need to find the image of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ image is all vectrs of the form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (x_1 + x_2 + x_3)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

any scalar multiples of the vectr $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ line in $\mathbb{R}^3$

23 Reflectn about line $y = x/3$ in $\mathbb{R}^2$.
the line L is given by $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \rightarrow \sqrt{1/9 + 1}$ $\sqrt{\frac{10}{9}}$ $= \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

the matrix of this projectn. $\begin{bmatrix} 1/10 & 3/10 \\ 3/10 & -1/10 \end{bmatrix}$

so the kernel is $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}_{R_2 - 3R_1} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

the kernel is $\vec{0}$, the image is $\vec{e}_1$ & $\vec{e}_2$

30. so $im(A) = \left\{\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right\} \rightarrow \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$

38. ~~let~~ A be a square matrix

$\ker(A)$ + ~~Ker~~ A^2 If there is a vector $\vec{x}$ that solves $A\vec{x}=0$,
then $A^2\vec{x} = A(Ax) = A0 = 0$. But A^2 is a different matrix than A, so
we can only say that $\ker(A) \subset \ker(A^2)$

b. The opposite holds for $\mathrm{im}(A)$, $\mathrm{im}(A^2)$, $\mathrm{im}(A^3)$
$\mathrm{rank}(A^2) + \mathrm{null}(A^2) = n$ (rank-nullity theorem)
$\mathrm{rank}(A) + \mathrm{null}(A) = n$

From above, $\mathrm{null}(A) \le \mathrm{null}(A^2)$, so
$\mathrm{rank}(A) + \mathrm{null}(A) = \mathrm{rank}(A^2) + \mathrm{null}(A^2) \geq \mathrm{rank}(A^2) + \mathrm{null}(A)$
$\Rightarrow \mathrm{rank}(A) \geq \mathrm{rank}(A^2)$

49. let $\vec{x}$ + $\vec{y}$ be in $\ker(A)$.
consider $k\vec{x} + l\vec{y}$ ~~then~~ $A(k\vec{x}+l\vec{y}) = A(k\vec{x}) + A(l\vec{y})$
$= kA\vec{x} + lA\vec{y} = 0$

So it is closed under addition/ scalar mult.

3.2

16. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}6\\5\\4\end{bmatrix}$

$$\begin{bmatrix}1&3&6\\1&2&5\\1&1&4\end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix}1&3&6\\0&-1&-1\\0&-2&-2\end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix}1&3&6\\0&-1&-1\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&3&6\\0&1&1\\0&0&0\end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix}1&0&3\\0&1&1\\0&0&0\end{bmatrix}$$

$c_1 = 3$ so.
$c_2 = 1$ $3x_1 + x_2 = x_3$
c_3 anything

x_3 is redundant, and not linearly independent

37 Let $\{\vec{v}_i\}$ be linearly independent. And let T be a linear transformation from $\mathbb{R}^n \to \mathbb{R}^p$

Are the vectors $T(\vec{v}_1), T(\vec{v}_2) \cdots T(\vec{v}_m)$ necessarily linearly independent? How can you tell?

No $\rightarrow$ one way you can tell is to see if the kernel of the matrix formed by the linear transformation only contains $\vec{0}$.

42 c Let $\{\vec{v}_i\}$ be a set of perpendicular unit vectors in $\mathbb{R}^n$. Show that these vectors are lin. ind.

They are lin. ind if $\ker\left(\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_m\end{bmatrix}\right) = \{0\}$,

or the only solution to

$$\sum_{i=1}^{m} c_i \vec{v}_i = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m = 0 \quad \text{is} \quad c_i = 0$$

take the dot product $\vec{v}_i$ on both sides

$$\left\langle \vec{v}_i \ \sum_{i=1}^{m} c_i \vec{v}_i \right\rangle = \langle \vec{v}_i, 0 \rangle$$

$$= c_i \langle \vec{v}_i, \vec{v}_i \rangle = 0$$

$$\Rightarrow c_i = 0.$$

this holds for all $i=1$ to m, so

$$\ker\left(\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_m\end{bmatrix}\right) = \{0\} \Rightarrow \text{linearly independent}$$

45. Are the columns of an invertible matrix linearly independent?

Yes - since it is invertible, then $\ker(A) = \{0\}$, or the linear combination of the columns equal to the 0 vector is $\{\vec{c}_i\} = 0$, so that implies linear independence.

3.3 6, 25, 29, 56, 57

6 $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ $\quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so

a basis of the image is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

and thus

$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 0$, so the basis for the kernel is

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$ $\quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\uparrow \quad \uparrow \quad \uparrow$

a basis for the image is given by the

span $\left(\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right)$ and since

$x_1 + 2x_2 + 5x_4 = 0$

$x_3 - x_4 = 0.$

$v_2 = 2v_1$

$v_4 = 5v_1 - v_3$

a basis for the kernel is given by

span $\left(\begin{bmatrix} 5 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$

29. A basis for the subspace of $\mathbb{R}^3$ defined by

$$2x_1 + 3x_2 + x_3 = 0$$

is $x_3 = s$, $x_2 = t$ so $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$

$$= -\frac{3}{2}t - \frac{1}{2}s$$

so the basis is:

$$s\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}$$

spanned by $\left(\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}\right)$

56. Consider a nilpotent matrix of order m.

show that $\{\vec{v}, A\vec{v}, A^2\vec{v}, \ldots A^{m-1}\vec{v}\}$ are lin. ind.

consider $c_0\vec{v} + c_1 A\vec{v} + \ldots c_{m-1}A^{m-1}v = 0$

$$\sum_{i=0}^{m-1} c_i A^i\vec{v} = 0.$$

multiply by A^{m-1}

so $\sum_{i=0}^{m-1} c_i A^{m-1+i}\vec{v} = 0$ $\quad A^{m-1+i}\vec{v} \Rightarrow A^{i-1}A^m\vec{v}$

$$\Rightarrow A^{i-1}0$$

$$= 0.$$

except when $c_0 A^{m-1}\vec{v} = 0$ $\quad A^{m-1}\vec{v} \neq 0$, so $c_0 = 0$.

repeat ~~m~~ each time multiplying by A^{m-2} this time

eventually get $c_i = 0$ $\forall i = 0$ to $m-1$.

so the set is lin. ind

57. Let m be the smallest integer such that $A^m = 0$.

There are m linearly independent vectors in $\mathbb{R}^n$, so $m \le n$

and $A^n = A^m A^{n-m} = 0 A^{n-m} = 0$.

3.4 11, 30, 33, 39, 52.

11. $\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$

so can $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ be solved? $\rightarrow$ yes

$c_1 = 1/2$ $c_2 = 1/2$ so

the coordinates of $[\vec{x}]_B = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

30. $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$$B = \begin{bmatrix} [T(\vec{v}_1)]_B & [T(\vec{v}_2)]_B & [T(\vec{v}_3)]_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

33. $T(\vec{x}) = (\vec{v}_2 \cdot \vec{x})\vec{v}_2$ $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ $\{\vec{v}_i\}$ is a unit vector.

this is a projection into the plane formed by $\{\vec{v}_i\}_{i=1}^{3}$

so the B matrix is $\begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) & T(\vec{v}_3) \end{bmatrix}$

$T(\vec{v}_1) = (\vec{v}_2 \cdot \vec{v}_1)\vec{v}_2 = 0$ $(\vec{v}_1 \perp \vec{v}_2)$

$T(\vec{v}_2) = (\vec{v}_2 \cdot \vec{v}_2)\vec{v}_2 = \vec{v}_2$

$T(\vec{v}_3) = (\vec{v}_2 \cdot \vec{v}_3)\vec{v}_2 = 0$ $(\vec{v}_2 \perp \vec{v}_3)$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is any basis of $\mathbb{R}^3$, pick $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

so $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

39. Find the basis $\mathfrak{B}$ of $\mathbb{R}^3$ such that $\mathfrak{B}$-matrix B of T is diagonal

Reflection about line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

If we can find a basis such that
$$T(v_1) = a v_1$$
$$T(v_2) = b v_2$$
$$T(v_3) = c v_3$$

then $[T(v_1)]_{\mathfrak{B}} = a$ $\quad [T(v_2)]_{\mathfrak{B}} = b$ $\quad [T(v_3)]_{\mathfrak{B}} = c$

so B matrix is $\begin{bmatrix} [T(v_1)]_{\mathfrak{B}} & [T(v_2)]_{\mathfrak{B}} & [T(v_3)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Pick $\vec{v}_1 \cdot \vec{v}_2 = 0$ then we'll have a basis for $\mathbb{R}_3$.
$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$

If we pick $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then the B matrix of the reflection is

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ so $b = -1$, $c = -1$. Let $v_2 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

so $\vec{v}_1 \cdot v_2 = v_x + 2v_y + 3v_z = 0$

$v_1 \times v_2 = \begin{bmatrix} (2v_z - 3v_y) \\ (3v_x - v_z) \\ (v_y - 2v_x) \end{bmatrix}$ Pick $v_x = 1$, $v_y = -1$, $v_z = -1$,

then $v_3 = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}$

so $V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $V_3 = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}$

the reflection matrix in the plane is

$$A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 & 2u_1u_3 \\ 2u_1u_2 & 2u_2^2 - 1 & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & 2u_3^2 - 1 \end{bmatrix} \qquad u = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and

$$\begin{array}{ccc} \vec{x} & \xrightarrow{A} & T(\vec{x}) \\ S^{-1}\downarrow\uparrow S & & \uparrow S \downarrow S^{-1} \\ [x]_B & \xrightarrow{B} & [T(x)]_B \end{array} \qquad S = \begin{bmatrix} 1 & 1 & -5 \\ 2 & 1 & 4 \\ 3 & -1 & -1 \end{bmatrix}$$

If $A = SBS^{-1}$, for any x, then it should work

So

52. If B is a basis of $\mathbb{R}^n$, is the transformation

$T(\vec{x}) = [\vec{x}]_B$ linear? Let $\{\vec{v}_i\}_{i=1}^n$ be a basis for $\mathbb{R}^n$

~~$T(\vec{x}$~~ these vectors span $\mathbb{R}^n$ and are linearly independent by definition

So $T(\vec{x}) = [\vec{x}]_B$ is just finding the coordinates $\vec{c}$ that solves

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_n\vec{v}_n$ or

$\vec{x} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} = A\vec{c}$ this can be solved since all the $\vec{v}_i$ are linearly independent and hence invertible, so $[\vec{x}]_B = \vec{c} = A^{-1}\vec{x}$.

since we know the matrix of this linear transformation, we're done. but lets verify Linearity:

so $T(\vec{x}+\vec{y})$ solves $\vec{x}+\vec{y} = \tilde{c}_1 \vec{v}_1 + \dots \tilde{c}_n \vec{v}_n \Rightarrow \tilde{c} = A^{-1}(\vec{x}+\vec{y})$

and $\tilde{c} = A^{-1}(\vec{x}+\vec{y})$

$= A^{-1}\vec{x} + A^{-1}\vec{y}$ (matrix properties)

$= c_{\vec{x}} + c_{\vec{y}}$ (linearity of coordinates)

so $[\vec{x}+\vec{y}]_B = [\vec{x}]_B + [\vec{y}]_B$

and pg 139 shows that $[k\vec{x}]_B = k[\vec{x}]_B$,

and we have shown

$T(\vec{x}+y) = T(\vec{x}) + T(\vec{y})$

$T(k\vec{x}) = kT(\vec{x})$, so the transformation is linear.