Math 2270-3
Spring 2005
March 10, 2005

Name: ____________________________________

Exam II

Instructions: This exam is a total of 120 points. Answer each question carefully and thoughtfully to receive full credit. Partial credit will be awarded, and points will be deducted if you write the answer down to a problem without justifying your steps. You do not need to simplify your answer unless it helps for clarity.

True/False (4 points each) Answer each question by marking "T" if the statement is true or "F" if the statement is false. **If the statement is false, then in the line below write the statement that would make it true OR provide a counterexample.**

F 1. The dimension of the space $\mathbb{R}^{2\times 3}$ is ~~5~~. 6

T 2. If the matrix A is orthogonal, then A^2 is orthogonal.

F 3. The formula $(AB)^T =$ ~~$A^T B^T$~~ $B^T A^T$ holds for all matrices A and B.

T 4. If the kernel of a linear transformation T from P_4 to P_4 is trivial, then T is an isomorphism.

F 5. If $\vec{x}$ and $\vec{y}$ are two orthogonal vectors in $\mathbb{R}^n$, then $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$.

Short Essay (20 points): 6. The majority of Chapter 5 addressed the idea of orthogonality. Why are orthonormal bases of a subspace V important? A successful essay is one that takes an idea/concept from orthogonality and expounds upon its virtues. Possible ideas to address in your essay could include (but are not limited to) (a) what orthonormal means, (b) how you would project a vector $\vec{x} \in \mathbb{R}^n$ onto a subspace V, (c) generating an orthonormal basis, (d) properties of orthogonal transformations.

Proofs (10 points each): Prove two of the following four statements. Feel free to answer them in any order.

7. Show for a matrix $A \in \mathbb{R}^{m\times n}$ that $\ker(A) = \ker(A^TA)$. Remember that you are proving equality of two sets.

8. Give an algebraic proof of the triangle inequality:

$$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Hint: Expand $\|\vec{x}+\vec{y}\|^2$ as $(\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$ and apply the Cauchy-Schwarz inequality.

9. If $\mathcal{B} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_m)$ is a basis for a subspace $V \subset \mathbb{R}^n$, then for $\vec{x}$, $\vec{y}$ in V,

$$[\vec{x}+\vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}},$$

where $[\vec{x}]_{\mathcal{B}}$ is the coordinate vector for $\vec{x}$.

10. Let $T(\vec{x})$ be an orthogonal linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are orthogonal, then so are $T(\vec{v})$ and $T(\vec{w})$.

7. Let $x \in \ker(A)$. Then $A\vec{x}=0$. So $A^TAx = A^T0 = 0$, and $x \in \ker(A^TA)$, so $\ker(A) \subset \ker(A^TA)$.
Let $x \in \ker(A^TA)$, so $A^TAx = A^T(Ax) = 0$. $Ax \in \text{im}(A)$, and $Ax \in \ker(A^T) = (\text{im } A)^\perp$, so it must be that $Ax = 0$ implying $x \in \ker(A)$, so $\ker(A) = \ker(A^TA)$

8. $\|x+y\|^2 = (x+y)\cdot(x+y) = \|x\|^2 + 2(x\cdot y) + \|y\|^2$
$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ (Cauchy Schwarz)
$\leq (\|x\| + \|y\|)^2$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

9. $[x]_{\mathcal{B}}$ means the vector $\vec{c}$ that solves $\vec{x} = B\vec{c}$, where $B = [v_1 \cdots v_m]$.
So $\vec{c} = B^{-1}\vec{x}$. Define $[y]_{\mathcal{B}}$ similarly.
So $[x+y]_{\mathcal{B}} = \vec{x}+\vec{y} = \tilde{c}_1 v_1 + \cdots \tilde{c}_n v_n = B\tilde{c} \Rightarrow \tilde{c} = B^{-1}(\vec{x}+\vec{y})$
$= B^{-1}\vec{x} + B^{-1}\vec{y}$
$= [x]_{\mathcal{B}} + [y]_{\mathcal{B}}$

10. If $\vec{v}$ + $\vec{w}$ are orthogonal, then

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2.$$

consider $\|T(\vec{v}) + T(w)\|^2 = \|T(v+w)\|^2$: linearity

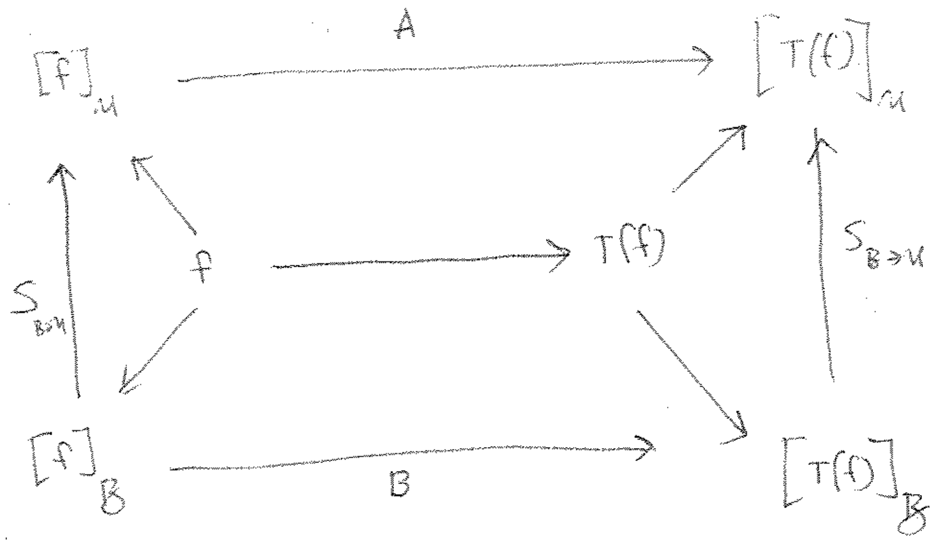
$= \|\vec{v} + w\|^2$ orthogonality of T

$= \|v\|^2 + \|w\|^2$ Pythagorean Thm

$= \|T(v)\|^2 + \|T(w)\|^2$ orthogonality of T

(10 points) 11. The following diagram summarizes the relationships between a linear space V and two different bases $\mathcal{U}$ and $\mathcal{B}$ of V, where we have:

$$\mathcal{B} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_m)$$
$$\mathcal{U} = (\vec{u}_1, \vec{u}_2, ..., \vec{u}_m)$$



Identify the columns in each of the matrices A, B, and $S_{\mathcal{B}\to\mathcal{U}}$, and how their columns would be found.

$A = \begin{bmatrix} [T(u_1)]_{\mathcal{U}} & [T(u_2)]_{\mathcal{U}} & \cdots & [T(u_m)]_{\mathcal{U}} \end{bmatrix}$

$B = \begin{bmatrix} [T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} & \cdots & [T(v_m)]_{\mathcal{B}} \end{bmatrix}$

$S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} [v_1]_{\mathcal{U}} & \cdots & [v_m]_{\mathcal{U}} \end{bmatrix}$

Part IV: (10 points each) Answer 3 of the following 6 questions. You may answer them in any order you wish.

12. Find the QR factorization of:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

13. Find an orthonormal basis of the following set of vectors:

$$\left(\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix} \right)$$

14. Find a basis for the space of all matrices in $\mathbb{R}^{2\times 2}$ where trace$(A) = 0$. What is its dimension?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

15. Determine if $T(f) = f'$ is an isomorphism with respect to the basis $V = (\cos t,\ \sin t)$. If T is not an isomorphism, find a basis for the image and the kernel of T.

16. Find a least squares solution to the equation $Ax = b$, where:

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

17. Find the matrix of the orthogonal projection for the set of vectors:

$$\left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)$$

12. $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$ (A is already upper triangular)

13. Use Gram Schmidt $\quad v_1 = [4\ 0\ 3]^T \quad v_2 = [5\ 8\ 0]^T$

$u_1 = \frac{1}{5}[4\ 0\ 3]^T \quad v_2^\perp = v_2 - (u_1 \cdot v_2)u_1$

$u_1 \cdot v_2 = \frac{20}{5} = 4$, so $[5\ 8\ 0]^T - \frac{4}{5}[4\ 0\ 3]^T = \frac{1}{5}[9\ \ 40\ -12]^T$

$\|v_2^\perp\| = \frac{1}{5}\sqrt{1825} = \sqrt{73}$, so $u_2 = \frac{1}{5\sqrt{73}}[9\ \ 40\ -12]^T$

14. If $A \in \mathbb{R}^{2\times 2}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a+d=0$, then $a=-d$.

So A is of the form $\begin{bmatrix} -d & b \\ c & d \end{bmatrix} = d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

basis: $\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$

Dimension: 3

15. This is a linear transformation. any element in V is expressed as

$x \in V \Rightarrow x = c_1 \cos t + c_2 \sin t.$ $[x]_V = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$T(x) = -c_1 \sin t + c_2 \cos t.$

so $[T(x)]_V = \begin{bmatrix} c_2 \\ -c_1 \end{bmatrix}$, and the B matrix is. $[T(x)]_V = B[x]_V$,

or $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ the rank of $B = 2$, so $\ker(B) = \{0\}$, and

thus $\ker(T) = \{0\}$. and it is an isomorphism

16. The least squares solution is the solution $x^* = (A^T A)^{-1} A^T b$.

$A^T A = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 50 & 41 \\ 41 & 38 \end{bmatrix}$ $(A^T A)^{-1} = \begin{bmatrix} .1735 & -.1872 \\ -.1872 & .2283 \end{bmatrix}$

$A^T b = \begin{bmatrix} 68 \\ 47 \end{bmatrix}$, so $x^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

17. The two vectors are orthonormal, so the matrix of the orthogonal projection is given by QQ^T,

which is $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$