Answer Key for Exam A

True/False (5 pts each)

Answer each question, marking "T" if the question is true and "F" if the question is false.

False — If a matrix A is in reduced row echelon form, then at least one of the entries in each column is a 1.

False — If A is a 5 by 6 matrix of rank 4, then the nullity of A is 1.

False — The formula $AB = BA$ holds for all invertible matrices A and B.

True — If $A^2 + 3A - 4I_3 = 0$ for a 3 by 3 matrix A, then A must be invertible. *Hint: factor*

False — A linear system with fewer equations than unknowns can have a unique solution.

Section 2. (10 points each)

Answer all of the following questions.

1. Find the inverse (if it exists) to the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Answer: Augmenting A to I_3 and subsequently inverting yields:

$$A^{-1} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ 3/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}$$

2. Multiply the following matrices:

$$\text{a. } \begin{bmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ -4 & 2 & 1 \end{bmatrix}^2$$

Answer: a. The product is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A side note to this problem is that two matrices can multiply together to an identity matrix even though neither matrix is invertible.

b. Note that to find the square of a matrix you do not square each element individually! By matrix-matrix multiplication, the answer is:

$$\begin{bmatrix} 1 & 8 & 2 \\ -4 & 11 & 4 \\ -8 & 16 & 3 \end{bmatrix}$$

3. Find a basis for the kernel and the image of the following matrix:

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 1 & -2 & 1 & 4 & -2 \\ 0 & 0 & -1 & -5 & 3 \end{bmatrix}$$

Answer: The reduced row echelon form of this matrix is:

$$rref(A) = \begin{bmatrix} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first and third columns contain leading ones, a basis for the image is given by:

$$image(A) = span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

Identifying the column vectors as v_i, then we see the following:

$$\begin{aligned} v_2 &= -2v_1 \\ v_4 &= -v_1 + 5v_3 \\ v_5 &= v_1 - 3v_3 \end{aligned}$$

So then the basis for the kernel is given by:

$$image(A) = span\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right)$$

4. Interpret the following linear transformation geometrically :

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hint: plot a few points and see what the transformation does to them.

Answer: If we take a vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

, our linear transformation transforms it into the vector

$$\begin{bmatrix} -y \\ x \end{bmatrix}$$

. What this does is flip the x and y components, which is reflection about the line $y = x$, and then takes the negative of each component. Doing this shows for a few examples shows that this is reflection about the line $y = -x$.

5. For which values k does the following system have a unique solution, no solution, or infinitely many solutions?

$$\begin{aligned} x + 5y &= 1 \\ -2x + ky &= 1 \end{aligned}$$

Answer: Writing the augmented matrix of this system:

$$\left[\begin{array}{cc|c} 1 & 5 & 1 \\ -2 & k & 1 \end{array}\right]$$

putting this in reduced row echelon form, we have:

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{k-5}{k+10} \\ 0 & 1 & \frac{3}{k+10} \end{array}\right]$$

We see that there will be no solution when $k = -10$ (the augmented matrix becomes undefined). There will never be infinitely many solutions, and a unique solution whenever $k \neq -10$.

Section 3. Proofs (10 pts each)

Answer 2 of the following 3 questions.

6. Prove that if $\text{rank}(A) = n$, where $A \in \mathbb{R}^{n\times n}$, then $A\vec{x} = 0$ has only the solution $x = 0$.

   Answer: If the rank of an n by n matrix is n, then according to the rank plus nullity theorem, the dimension of $\ker(A) = 0$, which means that 0 is the only vector in the kernel.

   Alternatively, since $\text{rank}(A) = n$, the $\text{rref}(A) = I_n$, so the augmented matrix $Ax = 0$ will have no free variables, implying that only 0 solves $Ax = 0$.

7. Let T be a transformation from $\mathbb{R}^m \to \mathbb{R}^n$, Prove that for $\vec{v}$, $\vec{w} \in \mathbb{R}^m$, k a scalar, if

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$$
$$T(k\vec{v}) = kT(\vec{v}),$$

   then T is a linear transformation from $\mathbb{R}^m \to \mathbb{R}^n$. *Hint: What does it mean to be a linear transformation? What do you need to find?*

   Answer: T will be a linear transformation if there exists a matrix $A \in \mathbb{R}^{n\times m}$ such that $T(\vec{x}) = A\vec{x}$. Let $\vec{x} \in \mathbb{R}^m$. We know that we can write $\vec{x}$ as a linear combination of the basis vectors $\vec{e}_i$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + ... + x_m\vec{e}_m$$

   Now compute $T(\vec{x})$ using our assumptions:

$$\begin{aligned} T(\vec{x}) = T &\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \\ &= T\left(x_1\vec{e}_1 + x_2\vec{e}_2 + ... + x_m\vec{e}_m\right) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + ...T(x_m\vec{e}_m) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + ...x_mT(\vec{e}_m) = A\vec{x} \end{aligned}$$

   The last statement is just matrix-vector multiplication, and the matrix A has $T(\vec{e}_i)$ in each of its columns.

8. Consider perpendicular unit vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ in $\mathbb{R}^3$. Show that these vectors are linearly independent. *Hint: Form the dot product of $\vec{v}_i$ on both sides of the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$. How can you use this to show linear independence?*

Answer: If we take the dot product of the left hand side with any v_1, since the dot product is distributive, and $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$ and 1 otherwise, we get that the only coefficients c_i that solve our equation are $c_1 = c_2 = c_3 = 0$. Effectively this says that the kernel of the matrix formed with these vectors as column vectors is 0, so these vectors are linearly independent.

A lot of people said that the three vectors were the standard basis vectors, and then proved the statement for the specific example by arguing that the standard basis vectors do not have a common component. While this may conceptually help you see how to solve a problem, note that the proof does not specify what $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are. For example, the following set of vectors:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

have some common components but are still linearly independent.

Section 4. (20 pts)

Answer one of the following two questions.

9. In the first homework set you examined the procedure to fit a cubic function through a set of data points (x_i, y_i). This problem looks at *piecewise linear* interpolation rather than piecewise cubic.

   Assume that between each data point (x_i, y_i) and (x_{i+1}, y_{i+1}) we can fit a linear function $f_i(x) = a_i + b_i x$. So at each data point (x_i, y_i), the functions $f_i(x)$ and $f_{i-1}(x)$ must agree, that is:

$$y_i = a_i + b_i x_i$$
$$y_i = a_{i-1} + b_{i-1} x_i,$$

   for $i = 0 \ldots n$. Thus we have $(n+1)$ data points, and for each data point except (x_0, y_0) and (x_n, y_n) we have 2 conditions to satisfy. The goal of this problem is to determine all the a_i and b_i that give you a piecewise continuous linear function. So we have a total of $2(n-1) + 2 = 2n$ total conditions, and $2n$ unknowns. (We seek to find the coefficients for n functions $f_i(x)$, and each function has two unknowns a_i and b_i). Theoretically we should be able to solve this problem as a linear system.

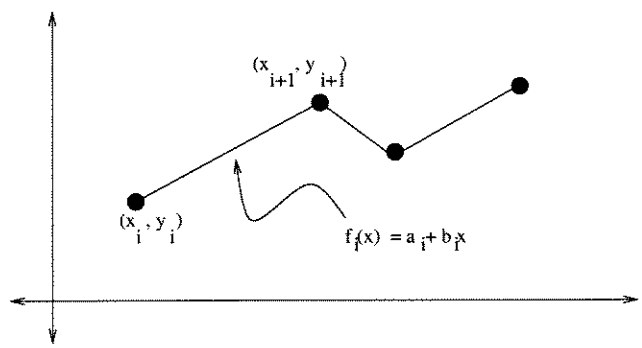

Figure 2: Example of piecewise linear functions fit through data points.

   a. How can you represent this problem as a linear equation? It may help to work out the simple case, using Figure 1 as a guide, and then extend it to n data points. What are the unknowns and what is the matrix that we multiply them by? Notice its unique structure.

   b. Now that you have found the matrix of the linear system, what is the inverse of that matrix? The following fact about partitioned matrices may be helpful:

$$\begin{bmatrix} A_0 & 0 & \cdots & \cdots & 0 \\ 0 & A_1 & 0 & \cdots & \vdots \\ 0 & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & A_n \end{bmatrix}^{-1} = \begin{bmatrix} A_0^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & A_1^{-1} & 0 & \cdots & \vdots \\ 0 & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & A_n^{-1} \end{bmatrix},$$

   provided each A_i is invertible. Find the inverse of each matrix A_i, effectively solving the problem.

Answer: This can easily be found by writing out each of the conditions that the linear function has to satisfy:

$$\begin{aligned} a_0 + b_0 x_0 &= y_0 \\ a_0 + b_0 x_1 &= y_1 \\ a_1 + b_1 x_1 &= y_1 \\ a_1 + b_1 x_2 &= y_2 \\ &\vdots \\ a_n + b_n x_{n+1} &= y_{n+1} \end{aligned}$$

The unknowns are the coefficients a_i and b_i. We will have a linear system $Ax = y$ if we use partitioned matrices. In this case, x represents the matrix:

$$x = \begin{bmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \\ \vdots \\ a_n \\ b_n \end{bmatrix}$$

The coefficient matrix will be diagonal with each entry A_i on the diagonal to be the matrix:

$$A_i = \begin{bmatrix} 1 & x_i \\ 1 & x_{i+1} \end{bmatrix}$$

Using the fact given, the inverse of each A_i is:

$$A_i^{-1} = \frac{1}{x_{i+1} - x_i} \begin{bmatrix} x_{i+1} & -1 \\ -x_i & 1 \end{bmatrix}$$

This effectively solves the problem.

10. Recall that the matrix for a rotations in the plane through an angle θ is of the following form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

a. What is the inverse of this transformation?

b. What is the matrix of the transformation of a rotation through an angle $-\theta$?

c. Show and explain in terms of the geometry why the answer you have in Part b is the same as Part a.

Answer: Using the fact that $\cos^2\theta + \sin^2\theta = 1$, the inverse of the transformation matrix is:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin\theta$, so a rotation through an angle $-\theta$ represents the inverse transformation found above. Geometrically this should make sense: if you rotate by an angle θ, undoing it (or finding the inverse) is rotation through an angle $-\theta$.