Problem 4.3.8 & 4.3.7 & 4.3.42

Find the matrix of the transformation

$$T(M) = M\begin{bmatrix}1&2\\0&1\end{bmatrix} - \begin{bmatrix}1&2\\0&1\end{bmatrix}M \quad \text{w/ respect to} \qquad T: U^{2\times 2} \to U^{2\times 2}$$

(space of upper triangular matrices)

$$\mathcal{U} = \left(\underset{\vec{u}_1}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix}0&1\\0&0\end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix}0&0\\0&1\end{bmatrix}} \right)$$

so if $M = \begin{bmatrix}a&b\\0&c\end{bmatrix}$, its coordinates are:

$$[M]_{\mathcal{U}} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

using Maple, $T(M) = \begin{bmatrix}0 & 2a-2c\\0&0\end{bmatrix}$ so

$$[T(M)]_{\mathcal{U}} = \begin{bmatrix}0\\2a-2c\\0\end{bmatrix}$$

and the matrix that takes $[M]_{\mathcal{U}} \to [T(M)]_{\mathcal{U}}$

is $A = \begin{bmatrix}0&0&0\\2&0&-2\\0&0&0\end{bmatrix}$

the kernel of this matrix is spanned by $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$

so T is <u>not</u> an isomorphism since $\ker(T) \neq \{0\}$

a basis for the kernel is $\operatorname{span}\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$, (we need to put $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ in the space of upper triangular matrices) and a basis for the image of A is spanned by $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

or span$\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right)$

For problem 7, we have the basis

$$\mathcal{B} = \Bigg(\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\vec{b}_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\vec{b}_2}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\vec{b}_3} \Bigg)$$

so we need to find the change of basis matrix S,

or $S_{\mathcal{U} \to \mathcal{B}}$:

so if $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $[M]_{\mathcal{U}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$M = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$$

$$= \begin{bmatrix} c_1 + c_3 & c_2 \\ 0 & c_1 - c_3 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

so
$$\begin{aligned} c_1 + c_3 &= a \\ c_2 &= b \\ c_1 - c_3 &= c \end{aligned} \quad \longrightarrow \quad \begin{aligned} c_1 &= \frac{a+c}{2} \\ c_2 &= b \\ c_3 &= \frac{a-c}{2} \end{aligned} \qquad [M]_{\mathcal{B}} = \begin{bmatrix} \frac{a+c}{2} \\ b \\ \frac{a-c}{2} \end{bmatrix}$$

so $[M]_{\mathcal{B}} = S\,[M]_{\mathcal{U}}$ by

$$S = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

[redacted]

The $\mathcal{B}$ matrix that takes

$$[T(M)]_{\mathcal{B}} = B[M]_{\mathcal{B}} \quad \text{is equal to:}$$

$$[T(M)]_{\mathcal{M}} = A[M]_{\mathcal{M}} \quad \text{and} \quad [M]_{\mathcal{B}} = S[M]_{\mathcal{M}}$$

$$\text{so } [T(M)]_{\mathcal{B}} = S[T(M)]_{\mathcal{M}},$$

$$B[M]_{\mathcal{B}} = SA[M]_{\mathcal{M}}$$

$$= \quad BS[M]_{\mathcal{M}} = SA[M]_{\mathcal{M}}$$

$$\text{So } \quad B = SAS^{-1}$$

using maple, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ is the matrix of the transformation.

the kernel of B is spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

which in our basis $\mathcal{B}$ is the the matrix

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (remember $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [M]_{\mathcal{B}}$ and

these correspond to matrices

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where

$$c_1 + c_3 = a$$
$$c_2 = b$$
$$c_1 - c_3 = c$$

the image of $\mathcal{B}$ is spanned by the vector $\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$, which

in our space is the matrix $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$,

so T is <u>not</u> an isomorphism.