Limit Review

This is a review sheet to remind you how to calculate limits. Some basic examples are sketched out, but for more examples you can look at Sections 9.1 and 9.2 in Harshbarger and Reynolds.

Limits

The definition of what it means for a function $f(x)$ to have a **limit** at $x = c$ is that: $\lim_{x\to c} f(x) = L$ (the limit of $f(x)$ as x approaches c equals L) exists if we can make values of $f(x)$ as close as we wish to L by choosing x sufficiently close to c. Otherwise, the limit does not exist.

While this is a good definition, for the purposes of this class we are more concerned with the practical aspects of applying this. To compute and understand limits, we need the notion of one-sided limits.

One-sided Limits

On graphs we can approach a point from two sides: either the left or the right. One-sided limits are the mathematical formalization of this idea.

- $\lim_{x\to c^-} f(x)$ is the limit as x approaches c from the *right*.
- $\lim_{x\to c^+} f(x)$ is the limit as x approaches c from the *left*

We say that the limit at $x = c$ exists if

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} = L = \lim_{x\to c} f(x)$$

In otherwords, the limits from the left and the right are the same. Figure 1 shows the difference between when a limit exists and when it doesn't. Note that it is not necessary for the function to be equal to its limit for the limit to exist. This discrepancy is rectified when we talk about continuity of a function.

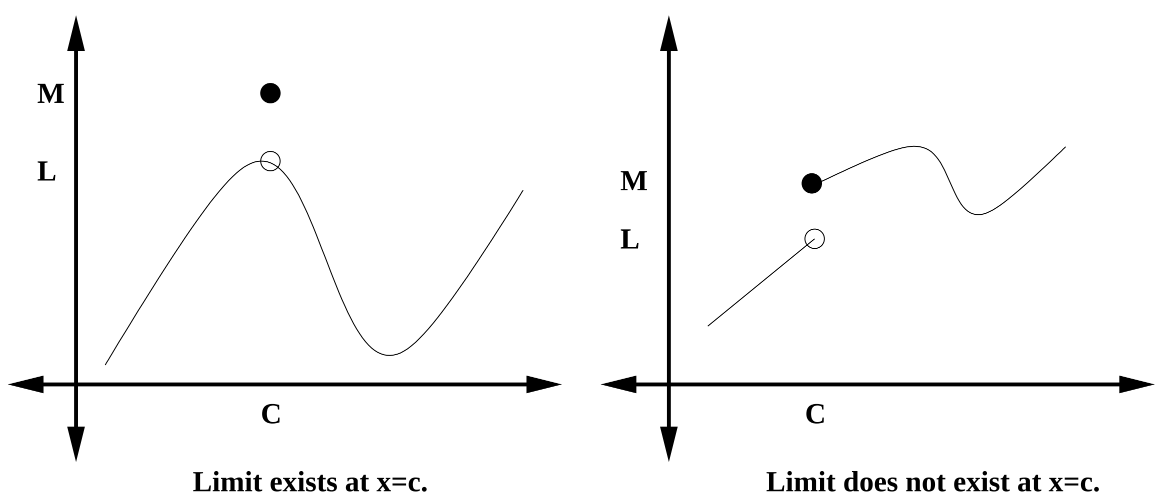

Figure 1: *Left*: The limit exists because the left and right limits are both L. Note that the value of the function at $x = c$ is not the same is the limit itself. *Right:* The limit at $x = c$ does not exist because the left and right side limits aren't equal

Calculating Limits

Graphically estimating limits becomes easier once the notion of one-sided limits is clear. If we are just given a function $f(x)$ without a graph of it, determining whether a limit exists at a point $x = c$ can be more challenging.

Algebraic Properties of Limits

There are several good properties of limits that can be proven, but here we just state as fact. First, let k be a constant, and let $f(x)$ and $g(x)$ be two different functions who have a limit L and M at $x = c$ respectively. The the following are true at $x = c$:

1. $\lim k = k$ (The limit of a constant function is just constant)
2. $\lim x = c$ (This can be shown by drawing this out)
3. $\lim f(x) \pm g(x) = L \pm M$ (The limit of the sum or difference is the sum or difference of the limits).
4. $\lim f(x)g(x) = LM$ (The limit of the product is the product of the limits)
5. $\lim \frac{f(x)}{g(x)} = \frac{L}{M}$ when $M \neq 0$. (The limit of the quotient is the quotient of the limits)
6. $\lim \sqrt[n]{f(x)} = \sqrt[n]{L}$ when $L > 0$.

The reason why these are good is that limits work just like algebra. In fact, for polynomial functions $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ or rational functions $h(x) = f(x)/g(x)$, these limits are easy to compute:

$$\lim_{x\to c} f(x) = f(c) \text{ when f(x) is a polynomial function}$$
$$\lim_{x\to c} \frac{g(x)}{h(x)} = \frac{g(c)}{h(c)} \text{ when h(c) is not 0}$$

The important point to note is that we can just "plug in" to evaluate the limit.
Example: Evaluate:

$$\lim_{x\to 5} \frac{x^2 - 4x}{x - 2} = \frac{5^2 - 4}{5 - 2} = \frac{21}{3}$$

$0/0$ Means "Do more math"

Example: Evaluate:

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \frac{2^2 - 4(5)}{2 - 2} = \frac{0}{0}$$

As you can see, we have a problem here in evaluation this limit at $x = 2$. Anytime you try the "plug-in" approach and get $0/0$ for a rational function,
you need to be more careful, thus "do more math." Essentially what you want to do is to simplify your function to understand the structure behind it. Ways you can do this are:

- Multiplying out an expression to see it it simplifies.
- Factoring to see if a common factor cancels.
- De-rationalizing the numerator or denominator if you have a square root.[1]

[1] We never discussed this case in class; hence don't worry about it. I just included it for completeness. If you are curious, we can talk about it.

Returning back to our example, it already seems to be multiplied out, so the tool we have at hand is to factor:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Evaluating Limits of the form $a/0$

If you try the plug-in approach and get something of the form $a/0$, where $a \neq 0$, then the function has a vertical asymptote at that point. The limit *may* exist if the one-sided limits are going to the same place (positive or negative infinity). Thus you need to evaluate the limit from the left or the right of the point in question. One of the ways to do this is to make a table of values, as we see in the following example.

Example: Evaluate:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2+3x+2}{x-1}$$

When we plug in $x = 1$, we get $6/0$. We can't factor this expression, so we must resort back to evaluating the one-sided limits. When we do this, we see:

x	f(x)	x	f(x)
0	-1	2	12
0.5	-7.5	1.5	17.5
0.9	-55.1	1.1	65.1
0.99	-595.01	1.01	605.01
0.999	-5995.001	1.001	6005.001
0.9999	-59999.0001	1.0001	60005.0001

So we see that as x approaches 1 from the left (columns 1 and 2), $f(x)$ seems to be going towards $-\infty$, and as x approaches 1 from the right (columns 3 and 4), $f(x)$ seems to be going towards ∞, so the one-sided limits are going to opposite places and hence the limit does not exist.

Limits of piecewise functions

A piecewise function is a function of the form:

$$f(x) = \begin{cases} g(x) & x > a \\ h(x) & x \leq a \end{cases}$$

If both $g(x)$ and $h(x)$ are polynomial functions, the only place that might trip you up is if we are evaluating the limit at $x = a$. If this is the case, then all you must evaluate the limits from the right and left side of a, as usual.

Limits at infinity

Evaluating a limit at positive or negative infinity is interesting because it tells us about the long term behavior of the function. However since infinity is more a concept than an actual number, we can take a limit from the left and right.

For rational functions, the idea is to compare the highest power (the order) of the numerator and the denominator and see who "wins out." In particular, we want to take advantage of the fact that:

$$\lim_{x\to\infty} \frac{x^m}{x^n} = \begin{cases} 0 & \text{when } m < n \\ DNE & \text{when } m > n \\ 1 & \text{when } m = n \end{cases}$$

The easiest way to evaluate a limit at infinity is to divide by the highest power in the rational function. Let's work an example:

Example: Evaluate

$$\lim_{x\to\infty} \frac{3x^2+5x^4+1}{9x^5-3x}$$

Since x^5 is the highest power, we will divide the numerator and denominator by that expression and simplify:

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2+5x^4+1}{9x^5-3x} &= \lim_{x\to\infty} \left(\frac{\frac{1}{x^5}}{\frac{1}{x^5}}\right)\left(\frac{3x^2+5x^4+1}{9x^5-3x}\right) \\
&= \lim_{x\to\infty} \frac{\frac{3x^2+5x^4+1}{x^5}}{\frac{9x^5-3x}{x^5}} \\
&= \lim_{x\to\infty} \frac{\frac{3x^2}{x^5}+\frac{5x^4}{x^5}+\frac{1}{x^5}}{\frac{9x^5}{x^5}-\frac{3x}{x^5}} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x^3}+\frac{5}{x}+\frac{1}{x^5}}{9-\frac{3}{x^4}} \\
&= \frac{0+0+0}{9-0} = 0
\end{aligned}$$