Math 1100-5
Fall 2004
September 21, 2004

Name: KEY

Name of person on your right: _____

Exam I

Instructions: This exam is a total of 100 points. Answer each question carefully and thoughtfully to receive full credit. Partial credit will be awarded, and points will be deducted if you write the answer down to a problem without justifying your steps. You do not need to simplify your answer unless it helps for clarity.

True/False (4 points each) Answer each question by marking "T" if the statement is true or "F" if the statement is false. If the statement is false, then in the line below write the statement that would make it true OR provide a counterexample.

T 1. The limit of $f(x)$ as x approaches c exists when both the limits from the left and the right of c are equal.

F 2. Rational functions are continuous for every real number. except where $h(x)=0$ if $f(x)=\frac{g(x)}{h(x)}$

T 3. The average rate of change of a function $f(x)$ from a to b is given by:

$$\frac{f(b)-f(a)}{b-a}$$

F 4. The derivative of $f(x)=\frac{g(x)}{h(x)}$ is ~~$f'(x)=\frac{g'(x)}{h'(x)}$~~ $f' = \frac{g'h - gh'}{h^2}$

F 5. If a function is ~~continuous~~ differentiable at $x=c$, then it is ~~differentiable~~ continuous at $x=c$.

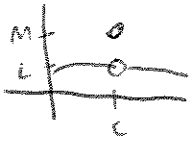
Short Essay (20 points): 6. State the definition of what it means for a function to be continuous at a point $x=c$. Then briefly explain why all parts of the definition are necessary to establish continuity. It may help to sketch a few examples to illustrate your point.

Defn of continuity: f is continuous at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

Need to have the limit exist Counter example:

$\lim_{x\to c} f(x)$ DNE, so it is discontinuous

and be the same as the value of the function itself otherwise



$\lim_{x\to c} f(x)$ exists, but $L \neq M$ and f is discontinuous

Limits/Derivatives (5 points each): Evaluate the following limits and derivatives.

7. $\lim_{x\to 1} f(x) = \begin{cases} x^2+9x-5 & x<1 \\ 3\sqrt{x}+1 & x\geq 1 \end{cases}$

$\lim_{x\to 1^+} f(x) = 3\sqrt{1}+1 = 4$

$\lim_{x\to 1^-} f(x) = 1+9-5 = 5$

Hence limit does not exist

8. $\lim_{x\to -5} \dfrac{3x^2+19x+20}{10-3x-x^2}$

of type $0/0$, so factor

$\lim_{x\to -5} \dfrac{(3x+4)(x+5)}{-(x+5)(x-2)} = \dfrac{3(-5)+4}{-(-5-2)} = \dfrac{-11}{7}$

9. $\lim_{x\to\infty} \dfrac{20x^2+5x}{x^3+4}$

divide top & bottom by x^3

$\lim_{x\to\infty} \dfrac{\frac{20}{x}+\frac{5}{x^2}}{1+\frac{4}{x^3}} \to \dfrac{0}{1} = 0$

10. $f(x) = \left(\sqrt{x}+4x^2\right)\left(x^{5/2}+3\right)$ Product rule

$f'(x) = \left(\frac{1}{2}x^{-1/2}+8x\right)\left(x^{5/2}+3\right) + \left(\sqrt{x}+4x^2\right)\left(\frac{5}{2}x^{3/2}\right)$

Quotient Rule

11. $f(x) = \dfrac{x^2+5x+1}{x+3}$

$f'(x) =$

$\dfrac{(2x+5)(x+3)-(x^2+5x+1)}{(x+3)^2}$

12. $f(x) = (x^4+1)^{-3}$ chain Rule

$f'(x) =$

$-3(x^4+1)^{-4}\cdot(4x^3)$

(10 points) 13. Use the definition of derivative to show that the derivative of x^2 indeed is $2x$. Let $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} =$$

$$\lim_{h \to 0} 2x + h = \boxed{2x = f'(x)}$$

(10 points) 14. Suppose that the revenue of a product is given by:

$$R(x) = 75x - x^2 \text{ for } 0 \leq x \leq 75,$$

where x represents the numbers of units sold, R is given in thousands of dollars.

a. Find the marginal revenue at $x = 10$.
b. What is the change in revenue from the 10^{th} to the 11^{th} unit sold?
c. Explain why Part b is a good approximation to the marginal revenue.

a. Marginal Revenue: $R'(x) = 75 - 2x$

$$R'(10) = 75 - 2(10) = 55$$

55 thousand dollars

b. Change in Revenue: $R(11) - R(10)$

$$R(11) = 75(11) - 11^2 = 825 - 121 = 704$$

$$R(10) = 75(10) - 10^2 = 750 - 100 = 650$$

$R(11) - R(10) = 54$ thousand dollars

c. B finds the average rate of change, close enough to $R'(10)$

(average rate of change approximates the derivative)

(10 points) 15. Find the equation of the tangent line of $f(x) = 2\sqrt{x} + 3x^2$ at $x = 4$. A graph of $f(x)$ and its tangent line is shown in Figure 1.

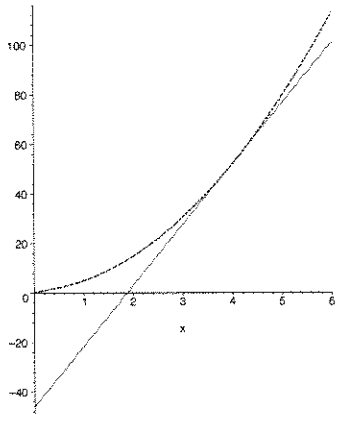


Figure 1: Graph of $2\sqrt{x} + 3x^2$ with its tangent line at $x = 4$

First, $f(4) = 2\sqrt{4} + 3(4^2) = 4 + 3(16) = 52$

$f'(x) = \frac{2}{2} x^{-1/2} + 6x = x^{-1/2} + 6x$

so $f'(4) = (4)^{-1/2} + 6(4) = 24.5$

equation of tangent line:

$y = 52 + 24.5(x-4)$

$= 24.5x - 46$