Finite Difference Methods for Parabolic Equations

In this note, we will briefly describe some highlights of finite difference methods for parabolic equations. As with any time evolution equations, the paramount concern for finite difference schemes is the issue of stability, where we will focus most of our attentions.
The obvious model equation to consider is the standard heat equation:

$$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0, \tag{1}$$

with initial condition

$$u(x, 0) = u_0(x), \tag{2}$$

and boundary conditions $u(0, t) = \phi_0(t)$ and $u(1, t) = \phi_1(t)$, for $t > 0$. We use the notation $u_j^n \approx u(x_j, t_n)$, where $x_j = j\Delta x$ and $t_n = n\Delta t$. If we use vector notations, a finite difference scheme can be represented as

$$u^{n+1} = Au^n + k^n. \tag{3}$$

The first question is the order of the method, namely the order of the local truncation error (LTE), in terms of Δt and Δx. If you substitute the exact solution into Eq.(3), you will need to divide by a factor Δt to get the correct order of LTE. To avoid confusion and make it easy to remember, it's probably more convenient to start the finite difference equation in the form that closely matches the original equation, such as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}. \tag{4}$$

It is clear that the finite difference equation approximates the differential equation. Whether the finite *difference* solution approximates the *differential* equation solution is a different matter! If we work on the finite difference equation in this form, then there should be no ambiguity in the order of the LTE when the Taylor expansions are carried out and the differential equations are used.
To focus on stability of the finite difference equation, we can imagine two different solutions u and v of the same finite difference equation (due to different initial data) and study the evolution of the difference $e = u - v$. It is readily found from Eq.(3) that

$$e^{n+1} = Ae^n. \tag{5}$$

The source term and boundary terms are eliminated from the consideration if we follow this direction (this is why we made the homogeneous assumption in stability analysis). Now it is obvious that the stability rests on the norm of the matrix A.

As it turns out, as always, estimating the norm of a matrix is no easy task, especially when A is not explicitly given (think of the implicit methods, where $A = T^{-1}B$ for some T). In the textbook a point is made that the 2-norm is equivalent to the spectral radius only if the matrix A is normal (p.288). This is an important point to keep in mind, though in most cases we should just use the Fourier analysis as the first choice.
There are two ways to go for the Fourier approach. In the first way, you can just pretend that the finite difference solution has the form

$$u_j^n = \lambda^n e^{ikj\Delta x}, \tag{6}$$

and ask what λ will make it indeed a solution. As we showed in class, very often we end up with a simple expression and our job is to analyze what makes λ to satisfy

$$|\lambda| \leq 1. \tag{7}$$

This will give you the stability condition (in l_2).
The second way is more general and easy to accept (unlike the previous one, simple though it is, you wonder how general the assumption is). We treat the finite difference solution u_j as the Fourier coefficients for a function $\hat{u}(\theta)$:

$$u_j = \frac{1}{2\pi}\int_0^{2\pi} \hat{u}(\theta)e^{ij\theta}d\theta. \tag{8}$$

with the understanding that the Parsavel's relation holds:

$$||u||_2 = ||\hat{u}(\theta)||_2. \tag{9}$$

Therefore, in order to have $||u^{n+1}||_2 \leq ||u^n||_2$, it is sufficient to have $||\hat{u}^{n+1}||_2 \leq ||\hat{u}^n||_2$. So the analysis is to find the relation between $\hat{u}^{n+1}$ and $\hat{u}^n$ and then bound the 2-norm.
The bottom line is that Fourier transforms diagonalize the matrix operator so it becomes trivial to compute 2-norms.
When will we need to go back to estimate the norm of matrix A? Consider the problem of an unusual boundary condition, such as $\alpha u + \beta u_x = 0$. Can we still use Fourier analysis? If not, we will need to study how the matrix is modified (only the first or last row) depending on your discretization of the boundary condition.
Finally it comes to the **Lax equivalence theorem**. It roughly says that

$$\text{consistency} + \text{stability} = \text{convergence} \tag{10}$$

This settles our question brought up earlier about the difference between approximation of equations and approximation of solutions. As we see it, the numerical stability of a particular scheme plays the central role.
Another topic touched upon in class is the maximum principle. This is an important result and tool in classic parabolic PDE analysis but we are not sure if it will be carried over to our numerical solutions (even they converge).

It turns out that only few schemes preserve this property and there is a heavy price to pay (a restriction on μ), if you want this property in the solution. As an example, consider the variable coefficient problem:

$$u_t = (a(x)u_x)_x \tag{11}$$

where $a(x) \geq a_0 > 0$. Denote $a_j = a(x_j)$, consider the following two schemes:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{a_{j+1/2}(u_{j+1}^n - u_j^n) - a_{j-1/2}(u_j^n - u_{j-1}^n)}{(\Delta x)^2}, \tag{12}$$

and

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a_j \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + \frac{a_{j+1} - a_{j-1}}{2\Delta x} \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}. \tag{13}$$

We will see that the first one can easily satisfy the maximum principle if we require

$$\frac{\max_j a_{j+1/2} \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{14}$$

References

[1] K.W. Morton and D.F. Mayers, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, 1994.

[2] J. Strikwerda, *Finite Difference Schems and Partial Differential Equations*, Wadsworth & Brooks/Cole, 1989.