Chapter 9

Alpha Trading

By the name of the strategies, an alpha trading strategy is to select and trade portfolios so the alpha is maximized. Two important mathematical objects are factor analysis and exploration of mean reversion phenomena if any.

9.1 Principal Component Analysis for Portfolios

We explain the idea of PCA from a linear algebra point of view. The main idea is to introduce a collection of factors to explain the majority of the variance of the returns with two simplifications: (1) the number of factors should be much smaller compared to the number of assets in the portfolio; and (2) the factors are uncorrelated. In linear algebra terms, we are looking for the first few eigenspaces represented by an orthogonal basis.
Consider the return vector of assets in the portfolio:

$$\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ R_N \end{pmatrix}$$

where N is the number of stocks and it can be quite large.
The covariance matrix is represented by

$$V = \mathbb{E}\left[(\mathbf{R}-\boldsymbol{\mu})(\mathbf{R}-\boldsymbol{\mu})^T\right] = QDQ^T \tag{9.1}$$

Since V is positive definite, we can assume that we can find orthogonal matrix Q to diagonalize V and

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_N \end{pmatrix}$$

contains all the eigenvalues of V. We assume that they are ordered

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N.$$

This suggests that if we introduce $\boldsymbol{f} = Q^T(\mathbf{R} - \boldsymbol{\mu})$, we will have

$$\mathbb{E}[\boldsymbol{f}\boldsymbol{f}^T] = D$$

so these factors $\boldsymbol{f}$ are uncorrelated.
The hope is that there are only a few eigenvalues in D that are significant, meaning they dominate over the other eigenvalues. Suppose we have determined that only the first m eigenvalues contribution really matters, we can expect to replace D by

$$\tilde{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_m \end{pmatrix}$$

with m much smaller compared to N.
To see how the reduction is achieved, we write the columns of eigenvectors $Q = [Q_1, Q_2]$, where Q_1 contains the first m eigenvectors. We expect V to be approximated by

$$\tilde{V} = Q \begin{bmatrix} \tilde{D} & \\ & 0 \end{bmatrix} = [Q_1, Q_2] \begin{bmatrix} \tilde{D} & \\ & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} = Q_1 \tilde{D} Q_1^T \tag{9.2}$$

The fact that we only include the contribution from the first m eigenvectors suggest that the return vector can be mostly represented by vectors in the eigenspace spanned by these eigenvectors, plus an idiosyncratic noise

$$\mathbf{R}' = f_1 \boldsymbol{v}_1 + f_2 \boldsymbol{v}_2 + \cdots + f_m \boldsymbol{v}_m + \boldsymbol{\epsilon} = Q_1 \boldsymbol{f}' + \boldsymbol{\epsilon} \tag{9.3}$$

Here we have taken the mean out of $\mathbf{R}$ to focus on the variance. In that case,

$$\mathbb{E}\left[\mathbf{R}'\mathbf{R}'^T\right] = Q_1 \mathbb{E}\left[\boldsymbol{f}'\boldsymbol{f}'^T\right] Q_1^T = Q_1 \tilde{D}\, Q_1^T \tag{9.4}$$

This means that the effect of replacing $\mathbf{R}$ by $\mathbf{R}'$ is that D is replaced by $\tilde{D}$, which has much smaller size and the factors $\boldsymbol{f}$ are all uncorrelated.

9.2 Statistical Arbitrage

Features:

- Trading signals are systematic, or rule-based, rather than driven by fundamentals;
- Trading book is market-neutral, the return of the total portfolio has zero beta with R_M;
- Mechanism for generating excess returns α is statistical;

- Diversification across stocks, low-volatility investment uncorrelated with the market.

The original idea is the so-called "pair-trading" strategy, that is, find a pair of stocks in the same industry or have similar characteristics and bet on the difference. It is believed that with the right combination (choice of the pair and proportion of each stock) the net beta can be made zero, and there may be a mean reversion mechanism to be found. Once we find the mean reversion mechanism, the simple "buy low and sell high" strategy can be made to work! This strategy has been developed into a general class of "long-short" strategies, such as the "130-30" fund that became famous.
The "generalized pair-trading includes a collection of "pairs", each is a pair between a stock with the industry index (ETF). The net result is that the holding in the ETF may represent a rather small fraction of the portfolio. The success of a statistical arbitrage scheme would require that there is a sufficiently large collection in the portfolio.
To obtain a market-neutral portfolio, we can use the factor model

$$R_i = \alpha_i + \sum_{j=1}^{m} \beta_{ij} f_j + \epsilon_i \tag{9.5}$$

and the return of the portfolio

$$R_P = \sum_{i=1}^{N} w_i R_i$$

If we choose the weights such that

$$\sum_{i=1}^{N} \beta_{ij} w_i = 0.$$

Then we have

$$R_P = \sum_{i=1}^{N} w_i \alpha_i + \sum_{j=1}^{m} \left[\sum_{i=1}^{N} w_i \beta_{ij} \right] f_j + \sum_{i=1}^{N} w_i \epsilon_i = \sum_{i=1}^{N} w_i \alpha_i + \sum_{i=1}^{N} w_i \epsilon_i \tag{9.6}$$

So a market-neutral portfolio is affected only by idiosyncratic returns.
Using the PCA developed in 9.1, we can use the eigenvectors of V to construct so-called eigenportfolios $\boldsymbol{w}^{(j)}, j = 1, 2, \ldots, N$ in which

$$w_i^{(j)} = \frac{v_i^{(j)}}{\sigma_i}$$

where $\boldsymbol{v}^{(j)} = (v_1^{(j)}, v_1^{(j)}, v_2^{(j)}, \ldots, v_N^{(j)})$ is the eigenvector corresponding to eigenvalue λ_j. The return of the eigenportfolio is

$$F_j = \sum_{i=1}^{N} w_i^{(j)} R_i = \sum_{i=1}^{N} \frac{v_i^{(j)}}{\sigma_i} R_i, \quad j = 1, 2, \ldots, m \tag{9.7}$$

In matrix form,

$$F_j = \mathbf{R}'^T \boldsymbol{w}^{(j)} = \mathbf{R}'^T \left(D^{-\frac{1}{2}} \boldsymbol{v}^{(j)} \right)$$

We can show that the correlation of F_j and F_j' vanishes for $j \neq j'$.
Now consider a stock portfolio (S) vs. the index fund (I) and we assume that their returns are related in the following way

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \beta \frac{dI(t)}{I(t)} + dX(t) \tag{9.8}$$

we will assume that α is negligible compared to the difference $X(t)$, which is modeled by

$$dX(t) = \kappa(m - X(t))\, dt + \sigma\, dW(t)$$

The solution to the above SDE is

$$X(t + \Delta t) = e^{-\kappa \Delta t} X(t) + (1 - e^{-\kappa \Delta t}) m + \sigma \int_t^{t+\Delta t} e^{-\kappa(t+\Delta t - s)}\, dW(s) \tag{9.9}$$

We have the equilibrium normal distribution for $X(t)$

$$\mathbb{E}[X(t)] = m, \quad \mathrm{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \tag{9.10}$$

Here κ is called the speed of mean-reversion and

$$\tau = \frac{1}{\kappa}$$

represents the characteristic time scale for mean reversion.
To implement a signal system to allow buy and sell strategies, we need to create signal definitions. In this strategy, we will use

$$s = \frac{X(t) - m}{\sigma / \sqrt{2\kappa}} \tag{9.11}$$

and the rational is that s measures the departure of X from its mean, relative to the normal fluctuation $\sigma/\sqrt{2\kappa}$. Once s is too large or too small, it suggests trading opportunities.
At each time during the trading period, we should monitor the s value. But how do we calculate s as it includes several parameters? We need to estimate these parameters based on the data collected. In the following, we describe briefly the procedure to estimate parameters.

1. α and β:

   We can view Eq.(9.8) in the discrete form

   $$R_n^S = \beta_0 + \beta R_n^I + \epsilon_n, \quad n = 1, 2, \ldots, 60.$$

   The parameters β_0 and β will just the y-intercept and the slope of the line that best fits the data points (R_n^I, R_n^S). Once we have β_0, we can obtain

   $$\alpha = \frac{\beta_0}{\Delta t} = 252 \cdot \beta_0$$

2. OU parameters:

   We consider a discrete version of Eq.(9.9):

$$X_{n+1} = a + b\,X_n + \eta_{n+1}, \quad n = 1, ..., 59.$$

   According to Eq.(9.9), we have

$$\begin{aligned} a &= m(1 - e^{-\kappa \Delta t}) \\ b &= e^{-\kappa \Delta t} \\ \mathrm{Var}(\eta) &= \sigma^2 \frac{1 - e^{-2\kappa \Delta t}}{2\kappa} \end{aligned}$$

   Again, a regression procedure can estimate the parameters a, b and $\mathrm{Var}(\eta)$. Once we have them, we can solve to obtain

$$\begin{aligned} \kappa &= -252 \log b \\ m &= \frac{a}{1-b} \\ \sigma &= \sqrt{\frac{\mathrm{Var}(\eta) \cdot 2\kappa}{1 - b^2}} \end{aligned}$$

Bibliography

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