Chapter 9

Portfolio Risk Management and Linear Factor Models

9.1 Portfolio Risk Measures

There are many quantities introduced over the years to measure the level of risk that a portfolio carries, and each has its own special emphasis. Here we list several of them.

A. Volatility

The volatility of a stock return is the most common measure to describe the risk level. It is defined as the standard deviation of a return per unit time. When a quote of volatility σ is given, we can expect the variance of the return over certain time period Δt to be $\sigma^2 \Delta t$. If we further assume that the return has normal distribution, we can even say that the probability for the return to be within $(\mu\Delta t - \sigma\sqrt{\Delta t}, \mu\Delta t + \sigma\sqrt{\Delta t})$ is about 0.68, where $\mu\Delta t$ is the expected return over the period Δt. The disadvantage of this measure is that there is no way to distinguish between up moves and down moves. In reality, many investors view the risk as adverse movements and this is not reflected well in the volatility measure.

B. Sharpe Ratio

This quantity measures the reward for taking risk, and it is defined as

$$S(P) = \frac{\mathbb{E}\left[R_P\right] - r}{\sigma_P}$$

The larger the Sharpe ratio, the more reward the investor will receive when the same amount of risk level is taken. For example, suppose portfolio A has Sharpe ratio 1, while portfolio B has Sharpe ratio 1.2. If their volatilities are the same at 20%, and the risk-free rate is 2%, then the expected return of portfolio A will be 22% and the expected return of portfolio B will be 26%. If we look at the Markowitz bullet and consider portfolios consisting of the

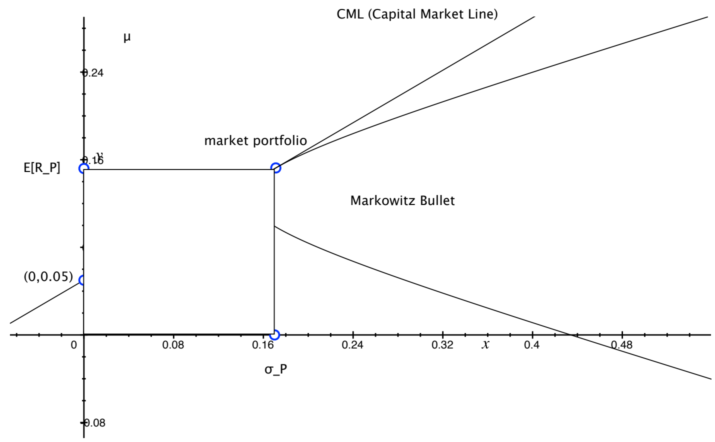

Figure 9.1: Sharpe ratio of the market portfolio as the slope of CML

risk-free asset and a risky portfolio on the frontier, then we find that those portfolios corresponding to the CML line will have the highest Sharpe ratio, which is the slope of the CML line.

In a leveraged fund, the Sharpe ratio stays the same, so the expected return can be estimated by the Sharpe ratio and the risk level that is taken.

Example 1 *Suppose the risk-free interest rate* $r = 4\%$*, and we have two portfolios with known expected returns and volatilities. We can calculate their individual Sharpe ratios and use them to make a decision as which portfolio is to be preferred. Portfolio 1 has higher expected return and comes with higher level*

	$\mathbb{E}[R]$	σ_P	$S(P)$
Portfolio 1	17 %	9%	1.44
Portfolio 2	15 %	5%	2.2

of risk as expected. Does that high risk level really justify, in light of another portfolio to be compared? One measure to look is the Sharpe ratio, in this case the portfolio 2 wins as its Sharpe ratio stands at 2.2*, substantially higher than* 1.44*, the Sharpe ratio of portfolio 1.*

C. Sortino Ratio

As we point out earlier, one disadvantage of the volatility measure is that it does not distinguish between the upward and downward moves. If only downward moves are counted as risk, we can introduce a modified fluctuation indicator that ignores upward moves. Suppose X_i is the observed return on day i, we introduce

$$Y_i = \begin{cases} a & X_i \geq a \\ X_i & X_i < a \end{cases}$$

and define the *downside sample semivariance* as

$$\sigma_a^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - a)^2 = \frac{1}{n}\sum_{i=1}^{n}\left[\min\{0, X_i - a\}\right]^2$$

Now if you want to count only downward moves, you can pick $a = 0$ in the above measure.

D. Maximum Drawdown

A drawdown is a streak of losses without substantial recovery, and it can be viewed as a peak-to-trough drop. Some investors are particularly sensitive to these occurrences and they fear the worst: a huge drawdown without a major break. One imagines that he/she bought the security at the peak and sold at the trough, which is the worst case that can happen to an investment. The maximum drawdown is like a worst case scenario, and many investors would particularly focus on them to see how much they can take. The exact definition for the maximum drawdown for a period $[0, T]$ is as follows

$$MDD(T) = \max_{0 \le u \le v \le T} (V(u) - V(v))$$

This definition involves taking maximum over two variables, and it can be awkward to work with. Actually it can be shown that it is equivalent to

$$MDD(T) = \max_{0 \le t \le T} (M(t) - V(t)),$$

where

$$M(t) = \max_{u \in [0,t]} V(u)$$

is the maximum value achieved by the time t.

E. Value-at-Risk (VaR)

VaR is probably the most widely used risk measure in the financial industry, and it gives the risk management a sense of damage control. The measure is better described using probability terms: Given $p \in (0, 1)$, the value-at-risk of a random variable X (it could be a return, a profit-and-loss, etc) for the level of probability p is denoted by

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \min\{x \mid F_X(x) \ge p\} = Q(p)$$

In another word, the p-VaR is the lowest p-quantile of X. Here $F_X(x) = \mathbb{P}\{X \le x\}$ is the CDF of the random variable X.

Notice that there are other versions of the VaR, and here we mostly use X as the gain/loss of certain portfolio. We use the following examples to illustrate the use and sense of a VaR quote.

Example 2 *Suppose we are told that a stock portfolio with a one-day p-VaR is \$20,000, with* $p = 1\%$. *The meaning of this quote is that the probability for the one-day loss* X *to go over \$20,000 is* 0.01.

Example 3 *In this example, we consider a single asset with initial value* $V = \$100,000$, *and an annual return denoted by the random variable* $R \sim N(0.14, 0.35^2)$ *(this says that* $\mu = 14\%$ *and* $\sigma = 35\%$*). With* $p = 1\%$, *what is the maximum loss at the end of the year?*

First we can look up the normal distribution table to find the 1%*-tile value* $Z_{1\%} \approx -2.33$. *For the distribution of* R, *it means*

$$\mathbb{P}\left[Z \leq -2.33\right] = \mathbb{P}\left[\frac{R - 0.14}{0.35} \leq -2.33\right] = \mathbb{P}\left[R \leq -0.6743\right] = 0.01$$

There is 1% *chance that the stock can lose more than* 67% *of its value. So the dollar amount of loss can be more than \$67,433.*

Methods to estimate VaRs can be categorized into the following:

- Historical methods: using data from the past;
- Variance-covariance methods;
- Monte Carlo simulation methods.

9.2 Introduction to Linear Factor Models

Suppose we need to model a collection of assets and it is natural to use a stochastic model such as the Black-Scholes for each asset. The random factors for these assets, more particularly the covariance matrix, will become the focus. Once the number of assets becomes large, the system will become more and more difficult to manage. In macroeconomical factor models, our economic understanding of the market suggests that we should be able to use some fundamental factors to explain a large part of the stock movements. On the other hand, in statistical factor models we use statistical methods to "discover" these important factors that are responsible for most of the movements.

Definition 1 *A (linear) factor model relates the return of an asset of a portfolio to the values of a limited number of factors, say* f_j, $j = 1, \ldots, m$

$$R_i = \beta_{i1} f_i + \beta_{i2} f_2 + \cdots + \beta_{im} f_m + e_i$$

where

- β_{ij}: *sensitivity of asset* i *to factor* j,
- e_i: *portion of the return that is not related to the* m *factors (idiosyncratic return)*

We then have the following questions (and answers):

- How can we estimate β_{ij}? (Linear regression).
- How do we model e_i? We let

$$e_i = \alpha_i + \epsilon_i, \quad \text{with } \mathbb{E}[\epsilon_i] = 0.$$

  Now we can write the linear factor model as

$$R_i = \alpha_i + \sum_{k=1}^{m} \beta_{ik} f_k + \epsilon_i$$

  For different assets i and i', we usually assume

$$\text{Cov}(\epsilon_i, \epsilon_{i'}) = 0, \quad \text{and } \text{Cov}(\epsilon_i, f_k) = 0.$$

What are these factors? If you work with a macroeconomical factor model, you could suggest some based on economic principles. In these statistical factor models, we aim at the following two aspects:

1. The number of factors (m) hopefully is not large.
2. The factors $\tilde{f}_j, j = 1, \ldots, m$ are uncorrelated:

$$\text{Cov}(\tilde{f}_j, \tilde{f}_{j'}) = 0 \;\text{ for } j \neq j', \quad \text{and} \quad \text{Cov}(\epsilon_i, \epsilon_{i'}) = 0 \;\text{ for } i \neq i',$$

Then we have the linear factor model

$$R_i = \alpha_i + \sum_{k=1}^{m} \beta_{ik} \tilde{f}_k + \epsilon_i$$

The coefficients β_{ik} are called factor loading coefficients. In matrix form we can write

$$\mathbf{R} = \boldsymbol{\alpha} + \boldsymbol{\beta}\, \mathbf{f} + \boldsymbol{\epsilon}$$

and the covariance matrix can be decomposed as

$$\text{Cov}(\mathbf{R}) = \beta \,\text{Cov}(\mathbf{f})\, \beta^T + \text{Cov}(\boldsymbol{\epsilon})$$

The significance of this decomposition can be explained as follows:

1. Factor models allow risk managers to perform risk management by approaching factors in a direct way.
2. We can allocate portfolios based on the risk profile.

9.3 Alpha and Beta

In a very simple form, we can write the return of an asset as

$$R = \alpha + \beta R_M + \epsilon$$

where R_M is the return of the market, and ϵ is the idiosyncratic return with mean zero. We can see that α is the part of the return in excess to the market, and β is the sensitivity to the market. If we have available data, we can use regression (such as least square fit to a line) to come up with a line in the (R_M, R) plane. This can be extended to several factors. One example of fundamental factor models is the Fama-French three-factor model, where the factors are

- market
- size of the company
- book/price ratio

Following CAPM, In this case we can write the model as

$$R - r = \alpha + \beta_1(R_m - r) + \beta_2(R_S - R_B) + \beta_3(R_H - R_L) + \epsilon$$

where the first factor refers the market return over the risk-free rate, the second refers the premium caused by small size vs big size of the company, and the third factor refers the premium generated by a high book/price ratio vs a low book/price ratio.