Chapter 7

Multi-factor Risk Modeling

In this chapter we study the problems arising from multiple risk factors, either through securities that derive their values from several other assets, or portfolios where with several assets included.

7.1 Real Options and Energy Derivatives

We introduce the concept of *real options*: the right to undertake certain business/investment projects sometime down the road. These options are often included in business/investment deals, and our goal of the study is to put a price tag on them. For example, an investor is considering investment in a 5-year real estate development project and he is concerned about the future risk. He would probably negotiate for a stipulation in the contract that would allow him to withdraw the whole or part of the investment 2 years from now. This additional right constitutes a real option, and the other party will assess its value so that they can add it to the cost. Unlike financial options, these options are not traded on the market so there is no market price to obtain. The valuation of these real options is a major topic in business school, and the only observation we can make at this stage is that the prices are estimated mostly by looking at similar projects and infer a proper value by comparisons.
Another example that arises in climate policies is the use of market to regulate pollution levels of certain industry. Here we consider a trading system that is called *cap-and-trade*, which was introduced in an effort to give incentives to the industry for reducing the total level of pollution. The system works like the following:

- A cap on total emission pollution is set by certain government agency;
- Certain amount of tradable units of permits are created and allocated to different companies in the industry;
- All companies that contribute to the emission pollution are allowed to trade the permits;

- At the end of the time period, each company will check their emission level and pay a penalty if the emission level exceeds the number of permits they own.

The idea is to use market efficiency to encourage the industry to invest in innovative technologies to reduce their emission levels.
We consider the following simple one-step binomial model:

At time $t = 0$, the permit price P_0 is observed, and we assume that the price P_1 at $t = 1$ has two possible values P_0u or P_0d, each with probability p and $1 - p$ respectively. Similarly the emission level of the company at $t = 0$ is Q_0 which is known, but the level at $t = 1$, Q_1 is unknown, believed to be either Q_0u or Q_0d, with probabilities q and $1-q$. At $t = 1$, the agency will check the emission level Q_1, together with its previous level Q_0, to see if they exceed the allowed level (permits purchased (X_0) combined with an initial endowment. If the pollution level is below what is allowed, the company does not pay any penalty, but the extra permits also expire with its value lost. If the emission level exceeds the allowance, the company will pay at the current price P_1, together with a penalty P, for every unit of emission exceeded. The decision to make at $t = 0$ is to purchase the right amount X_0 so that the allowance just matches the consumption. This can be translated to the following optimization problem:
Find the initial number of permits X_0 to purchase such that

$$F(X_0) = P_0 \cdot X_0 + (1 + r)^{-1}\mathbb{E}\left[g(X_0)^+ \cdot (P_1 + P)\right]$$

is at its minimum. Here

$$g(X_0) = Q_0 + Q_1 - X_0 - N$$

is the permit shortage (excess for negative values) at the end of the time period. For illustration purpose, the parameters are given as follows.

- $N = 50$ (tons) is the permit endowment;
- $P_0 = \$20$ is the permit price at $t = 0$;
- $Q_0 = 40$ (tons) is the emission level at $t = 0$;
- $P = \$100$ is the penalty for each permit shortage unit;
- $r = 6\%$ is the interest rate;

- $p = 0.8,\ q = 0.5$;
- $u = 1.2$ is the up factor, $d = 1/u$ is the down factor.

There will be four scenarios for P_1 and Q_1. We first consider the situation that price moves and emission level changes are independent. In this case, the expectation can be written as

$$\mathbb{E}\left[g(X_0)^+ \cdot (P_1 + P)\right] = [p(P_0 u + P) + (1-p)(P_0 d + P)] \cdot \\ \left[q(Q_0(1+u) - X_0 - N)^+ + (1-q)(Q_0(1+d) - X_0 - N)^+\right]$$

Since the objective function is not differentiable in X_0, it is necessary to use methods other than the critical point approach to find the minimum. It however, can be shown that the global minimum does exist.

7.2 Modeling of Multi-Asset Economy

We consider an economy with N securities, each with a price $S_i(t), i = 1, 2, \ldots, N$. For a time period $[0, T]$, we look at the returns

$$R_i = \frac{S_i(T) - S_i(0)}{S_i(0)}, i = 1, 2, \ldots, N$$

and they are considered random variables. The statistics of the economy is first summarized in the expected return and the covariance matrix of the returns:

$$\mathbb{E}[\mathbf{R}], \quad \mathbf{C}(\mathbf{R}) = [C_{i,j}]$$

where $\mathbf{R} = (R_1, R_2, \ldots, R_N)$, and $C_{i,j} = \mathrm{Cov}(R_i, R_j)$. Both of them can be estimated from data.
On the other hand, if we begin to model each security price:

$$\begin{aligned} \frac{dS_i}{S_i} &= \mu_i\, dt + \sum_{j=1}^{d} \sigma_{i,j}\, dW_j \\ &= \mu_i\, dt + \bar{\sigma}_i \cdot d\mathbf{W} \end{aligned}$$

where $\bar{\sigma}_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{id})$ and $\sigma_i = ||\bar{\sigma}_i|| = \sqrt{\sigma_{i1}^2 + \cdots + \sigma_{id}^2}$, we can make attempts to estimate the model parameters based on the observed expected returns and covariance matrix of returns.

7.3 Options on Several Underlyings

It is natural to consider options with payoffs dependent on several assets. We consider the following examples:

(A) Exchange Option

This is an option that allows the holder to exchange one share of stock 2 for one share of stock 1, at the expiration of the option. Obviously the holder will exercise the option only when the price of stock 2 is higher than the price of stock 1 at T. The payoff is therefore

$$V_T = (S_2(T) - S_1(T))^+$$

The price of the option can be written as

$$V_t = e^{-r(T-t)}\tilde{\mathbb{E}}_t\left[(S_2(T) - S_1(T))^+\right]$$

One is tempted to introduce one random variable for $S_2(T) - S_1(T)$ and the expectation becomes an integral with the pdf for this random variable. The problem is that there will be major issues in associating the statistics of the random variable with market information. Instead, if we use the stock price models in Section 7.2, $S_2(T)$ and $S_1(T)$ will have a joint lognormal distribution. A calculation of the double integral results in

$$V_t = S_2(t)N\left(d_+\left(\frac{S_2}{S_1}, T-t\right)\right) - S_1(t)N\left(d_-\left(\frac{S_2}{S_1}, T-t\right)\right)$$

where N is as usual the cumulative normal, and

$$d_+(x,\tau) = \frac{\log x + (r + \frac{1}{2}\nu^2)\tau}{\nu\sqrt{\tau}}, \quad d_-(x,\tau) = d_+(x,\tau) - \nu\sqrt{\tau}$$

and

$$\nu = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

(B) Choose max call: this is a call option that will allow the holder to choose the stock to be compared with K.

$$V_T = (\max(S_1(T), S_2(T)) - K)^+$$

(C) Spread option (Margrabe): this is an extension of the exchange option.

$$V_T = (S_2(T) - S_1(T) - K)^+$$

(D) Cross-currency options: here the exchange rate $X(t)$ at time t plays a major role.

1)
$$C_T = X(T)\left(S^f(T) - K_f\right)^+$$

2)
$$C_T = \left(X(T)S^f(T) - K\right)^+$$

3)
$$C_T = \bar{X}\left(S^f(T) - K_f\right)^+$$

4)
$$C_T = S^f(T)\left(X(T) - K\right)^+$$