Chapter 6

Credit Risk and Credit Derivatives

In today's business world, a credit event, such as a company's default or credit rating change (upgrade or downgrade), can lead to various consequences reaching all the way to unexpected remote corners of the world, and this is probably the work of so-called credit derivatives. One of the early motivations for credit derivatives is the need to address potential defaults of a company when the investor had a high stake in the company, probably through owning its corporate bonds, or in a swap contract. In this case a contract called credit default swap (CDS) is often used to protect the investor's interest and hedge the credit risk. But early financial innovators at that time never imagined the extensions of the idea and development of various financial products. They couldn't conceive today's widespread use of credit products as vehicles to speculate and make all kinds of bets on a wide range of events.

The presence of the credit risk is felt everyday on the financial markets, and it can be explained by the fact that corporate and many government bonds are traded at lower prices, or higher yields, compared to the US treasury bonds (assumed to be default-free) with the similar terms. The extra discount, or promised extra return, is a compensation for the risk taken by the investor: as there is a possibility that the investors may not get all their money back, and history is never short of catastrophes and they continue to appear in all different ways.

6.1 Credit Ratings of Companies

The creditworthiness of a company is assessed by one of those rating agencies, such as Moody's, S&P, and Fitch. Ratings are assigned by these rating agencies to corporate bonds as to help investors to estimate the likelihood of default, or a measure of the credit risk, over certain time period. Different rating agencies use different scales:

- Moody's: Aaa, Aa, A, Baa, Ba, ...

- S&P: AAA, AA, A, BBB, BB, ...
- Fitch's: similar to S&P.

Quite often an agent is under constraints set forth by the management in terms of what kind of corporate bonds he/she can purchase, such as the investment grades which usually require a Baa or higher rating. Central to all the ratings is the estimate of default probability, while each rating agency has its own method of estimation and the algorithms always take into account of a) historical data, and b) expectations of the company future businesses.

6.2 Default Probability and Survival Probability

The mathematical question to describe credit is how to quantify the risk of a default of a particular entity. We introduce the default probability of a company *by time* t as $Q(t)$, and the survival probability by t as $V(t) = 1 - Q(t)$. These are cumulative probabilities, and if we ask for the default probability over certain Δt we would have to look into the density. It is important to distinguish the conditional and unconditional probabilities in this case: the probability in question should be the probability of default over the next period of time $(t, t + \Delta t)$ *if the company has survived until* t. In another word, we need

$$
\begin{aligned}
&\mathbb{P}\left[\text{default over } (t, t+\Delta t),\ \text{given no default by } t\right] \\
=&\frac{\mathbb{P}\left[\text{default over } (t, t+\Delta t)\right]}{\mathbb{P}\left[\text{survival until } t\right]} \\
=&\frac{V(t) - V(t+\Delta t)}{V(t)}
\end{aligned}
$$

This should be proportional to Δt and it is natural to introduce a rate. In this case we have

$$
\boxed{\lambda(t) = -\frac{1}{V}\frac{dV}{dt}}
$$

as the **hazard rate**, or the **default intensity**. Intuitively speaking, the hazard rate measures the default rate in the next short period of time, given the survival for the time being. Alternatively, the survival probability and cumulative default probability can be written as

$$
V(t) = e^{-\int_0^t \lambda(s)\, ds} = e^{-\bar{\lambda}(t)\cdot t}
$$

where

$$
\bar{\lambda}(t) = \frac{1}{t}\int_0^t \lambda(s)\, ds
$$

is the average hazard rate over the period $(0, t)$.

6.3 Recovery Rates and Loss Distribution

When a company defaults, usually not all is lost so there is the question of how much can be recovered. The **recovery rate (RR)** gives the percentage of the face value the bond holders receive in the event of a default, and naturally it is uncertain so a random variable R is introduced to describe this percentage. Many studies have been devoted to this subject and there are some estimates based on historical data, and one of them, a Moody's statistics gives

$$RR \approx 59.1 - 8.356 \times \text{Default Rate}$$

One major question in this area is the correlation between the default probability and the recovery rate. We can argue that if defaults are more likely to occur then the loss can be potentially high, resulting in a low recovery. This claim that default probabilities and recovery rates are negatively correlated is actually supported by some empirical studies.

The **loss distribution** is a concept widely used in actuarial industry, as well as in defaultable bond pricing. It can have different specific definitions but in essence it describes the information combining default probabilities and the recovery rates given a default occurring. A generic notion is the probability of losing an amount lower than certain value by certain time. It is also similar and related to the concept of Value at Risk (VaR), which describes the threshold loss value in a portfolio with a probability p.

6.4 Estimation of Hazard Rates from the Market

Here is a toy example to illustrate the principle: suppose there is a 5-year zero-coupon corporate bond with face value \$1 issued by a company that matures in 5 years, but the company may default at time τ within the next 5 years. In that case the company will not be able to meet its obligation of fully paying back the principal amount. We assume that the distribution of τ is an exponential distribution with parameter λ, that is

$$Q(5) = \mathbb{P}[\tau \leq 5] = 1 - e^{-5\lambda}.$$

Suppose the risk-free interest rate (annualized) for the 5-year investment is r, and there is no payment in case of a default ($R = 0$), the discounted expected payoff at $T = 5$ is therefore

$$P = \mathbb{P}[\tau > 5] \cdot e^{-5r} + \mathbb{P}[\tau \leq 5] \cdot 0 = e^{-5(r+\lambda)}.$$

The yield of this bond is therefore $y = r + \lambda$, and we can see that the extra yield in addition to the risk-free rate is due to the constant intensity λ.

Now let us generalize this notion. Surely we cannot assume the default time to be an exponentially distributed random variable with a constant λ. However, for a given maturity T, we have the price of a defaultable bond in terms of an expectation of discounted payoff

$$P = e^{-yT} = V(T) \cdot e^{-rT} + Q(T)Re^{-rT}$$

Assuming small r and $\bar{\lambda}$, we have an estimate

$$\bar{\lambda}(T) \approx \frac{y-r}{1-R} = \frac{s}{1-R}$$

with s obviously defined as the yield spread. Suppose we have several bonds with maturities $T_1, T_2, \ldots, T_n$. For each one we can infer $\bar{\lambda}(T)$ from above so it is possible to fit a time-dependent function $\lambda(t)$ so that $\bar{\lambda}$ for those T values are matched. This is a boot-strapping procedure often used in these data analysis.

6.5 Real-World vs Risk-Neutral Probabilities

The default probability information gathered from the bond yields is necessarily risk-neutral as they are inferred from a market which is assumed to be complete. On the other hand, we can also estimate default probabilities from historical data sets, and derive so-called historical default probabilities. Typically the risk-neutral probabilities are higher than the historical probabilities of the same term, and the difference is called the "excess return". There are some explanations:

1. Corporate bonds are relatively illiquid so they are often sold at prices even lower relatively to what is implied from the actual expected loss;

2. Investors care particularly about those depression scenarios;

3. Defaults are often correlated: if one is triggered, it is more likely the others would follow.

6.6 Reduced Form Models

Here the focus is on modeling the default intensity λ, for which $\lambda\Delta t$ can be viewed as the likelihood of default over a next short period $(t, t+\Delta t)$, given that the company still survives at t. It is obvious that the intensity λ must be time-dependent to develop a particular default term structure. Not only λ should be time varying, but also it is most likely stochastic, thus opening the door for many models reflecting the forecast for the credit future of a particular entity, such as one or multi factors, mean reversion, and so on. In developing a practical model, a proper balance should be maintained between analytic tractability and comprehensive description of the phenomena.

6.7 Structural Models

The other type of models, pioneered by Merton (1973), is called structural model, and these models are based on the fundamental argument that a default is triggered when the company's total asset fails to meet the company's total liability. In another word, the total asset value of the company reached a level that is blow its total liability. Here we need to understand a basic corporate financing principle: a company usually has two ways raising capital - issuing debt or equity, if it needs to expand its business. The bond (debt) holders are capped at the gain, but they have the priority in collecting the remaining asset if the company goes bust.
Merton's original argument is the following. Let V be the total asset of the company and B the outstanding liability, and they are both time dependent. At maturity T,

- If $V > B$, the bond holders receive their promised amount B, while the equity holders take away the rest $(V - B)$;
- If $V \leq B$, which is the case where the company would fail its obligations, the bond holders take everything that is left (V), and the stock holders get nothing.

The payment received by the stock holders is therefore exactly the same payoff of a call option

$$(V - B)^+ = \begin{cases} V - B, & V > B \\ 0, & V \leq B \end{cases}$$

So for stock holders, they in fact have a call option on the value of the company (V), stuck at the face value of the debt (B), so the value of the stock $E(t)$ at $t < T$ can be expressed by the Black-Scholes formula (with a proper probability measure and a volatility that measures that future fluctuation of the company's value). At any time $t < T$, the value of the company is divided between the bond holders and the stock holders, meaning $V(t) = E(t) + B(t)$, so the value of the defaultable bond at t is

$$B(t) = V(t) - e^{-r(T-t)}\mathbb{E}\left[(V(T) - B)^+ | \mathcal{F}(t)\right]$$

This is one of the first clean expressions for modeling defautable bonds, and the dependence on the volatility as required in the Black-Scholes formula highlights the risk in the company's credit conditions.
One particular issue in using the Black-Scholes formula in the above valuation is the choice of volatility. In this case it is the volatility of the asset value $V(t)$. Where do we go to find that information? Suppose that the stock is traded and we observe some volatility of the stock price σ_E, but it is the volatility of the stock price, not the volatility of the asset σ_V. Fortunately, both the change in stock price and the change in asset value are driven by the same factor dW, we can use Ito's formula to conclude

$$\sigma_E E = \frac{\partial E}{\partial V}\sigma_V V = N(d_1(\sigma_V))\sigma_V V,$$

which is an equation for the unknown σ_V.
An obvious weakness of the model in this form is that defaults are allowed only at the maturity T. The extension of this model is to allow defaults to happen at the first time that V falls below the threshold. This leads to an application of the famous "first exit" problem. Let $V(t)$ be the value of the asset at time t, the first time V falls below B

$$\tau = \min\{s > 0 : V(s) \leq B\}$$

gives the default time, and the probability of default before t

$$Q(t) = \mathbb{P}[\tau < t]$$

as a function of time is the outcome of the model that can be calibrated to the market implied default probabilities. The problem becomes a calibration problem for the process $X(t) = V(t) - B$, that is the determination of the model parameters. In the case where X is a Brownian motion or Geometric Brownian motion, there are well-known results based on the reflection principle. We illustrate this approach by building a series of processes from the very basic.

1. $X(t) = W(t)$, with barrier $m > 0$. We define the exit time

   $$\tau_m = \min\{t : X(t) \geq m\}$$

   For any $w > 0$, the reflection principle gives

   $$\mathbb{P}[\tau_m \leq t, W(t) \leq w] = \mathbb{P}[W(t) \geq 2m - w]$$

   Therefore,

   $$\begin{aligned}
   \mathbb{P}[\tau_m \leq t] &= \mathbb{P}[\tau_m \leq t, W(t) \leq m] + \mathbb{P}[\tau_m \leq t, W(t) > m] \\
   &= 2\mathbb{P}[W(t) \geq m] \\
   &= \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}}\, dx = 2\left(1 - N\left(\frac{m}{\sqrt{t}}\right)\right).
   \end{aligned}$$

2. $X(t) = W(t),\ m \leq 0$. We define

   $$\tau_m = \min\{t : X(t) \leq m\}$$

   A similar calculation based on symmetry gives

   $$\mathbb{P}[\tau_m \leq t] = 2\left(1 - N\left(-\frac{m}{\sqrt{t}}\right)\right).$$

3. $X(t) = \sigma W(t),\ m > 0$.

   $$\tau_m = \min\{t : X(t) \geq m\}$$

   We have

   $$\mathbb{P}[\tau_m \leq t] = 2\left(1 - N\left(\frac{m}{\sigma\sqrt{t}}\right)\right).$$

4. $X(t) = \alpha + \sigma W(t)$ where α is a constant, we note that $X(t) \geq m$ is equivalent to $W(t) \geq \frac{m-\alpha}{\sigma}$ so
$$\mathbb{P}\left[\tau_m \leq t\right] = 2\left(1 - N\left(\frac{|m-\alpha|}{\sigma\sqrt{t}}\right)\right).$$
5. $X(t) = \alpha t + \sigma W(t)$ with constant $\alpha > 0$ and σ. This turns out to be a substantial problem that involves a change of measure:
$$\tilde{W}(t) = W(t) + \frac{\alpha t}{\sigma}$$
The calculation follows the change of measure formula
$$\mathbb{P}\left[\tau_m \leq t\right] = \mathbb{E}\left[I_{\{\tau_m \leq t\}}\right] = \tilde{\mathbb{E}}\left[\frac{1}{Z} I_{\{\tau_m \leq t\}}\right]$$

6.8 Adjusting Derivative Valuations for Counterparty Default Risk

When you entered a derivative contract with some counterparty, you may wonder what would happen if the counterparty is out of business. In case the derivative has negative value to you, there is practically no impact as the liquidators or the company that takes over will inherit that right and you just need to change the name of the payee when it comes to the payoff. It is going to be a problem, however, when the derivative has positive value to you as an asset, but suddenly nobody is ready to make that payoff to you. This is considered a counterparty credit loss and an adjustment is called for when you try to price it properly.
The principle in adjustment is that we should exclude the present value of the expected loss from the price. Consider the following example, where a derivative with expiration T without counterparty risk is quoted at f_0. It is assumed that defaults are possible at $t_1, t_2, \ldots, t_n = T$, with unconditional default probabilities $q_1, q_2, \ldots, q_n$ accordingly. If a default occurs at t_i, the loss would be $f_i(1-R)$ where f_i is the value of the derivative at t_i. Since the default probability at t_i is q_i, and the discounted expected value of f_i is f_0, the total expected loss is therefore
$$f_0 \sum_{i=1}^{n} q_i(1-R)$$
so the adjust price of the derivative should be
$$f_0^* = f_0 - f_0 \sum_{i=1}^{n} q_i(1-R)$$
If we can find a bond issued by this counterparty with the same maturity and its price is
$$B_0^* = B_0(1 - \sum q_i(1-R)), \quad B_0 = e^{-yT}, \; B_0^* = e^{-y^*T}$$

where y and y^* are the bond yields of the respective risk-free and defaultable bonds, we can write

$$f_0^* = f_0 e^{-(y^*-y)T}$$

Here is another example involving counterpart risk: a currency swap between a financial institution (A) and a counterparty (B) that is subject to default risk. In this currency swap, A pays a fixed interest rate r_A on a principal P_A paid in dollars to B, and receives a fixed rate r_B on principal P_B paid in Euro from B. Suppose that the swap is to last till T and interests payments are exchanged once a year at $t_1, t_2, \ldots, t_N = T$. The value of the swap at t_i is

$$P_A(1 + r_A) - P_B(1 + r_B)S(t_i)$$

where $S(t_i)$ is the dollar-Euro exchange rate at t_i. The principals are set up so that $P_A/P_B = S(0)$. The possible loss is therefore

$$\max\left(P_A(1 + r_A) - P_B(1 + r_B)S(t_i), 0\right) = P_B(1+r_B)\max\left(\frac{P_A(1 + r_A)}{P_B(1 + r_B)} - S(t_i), 0\right)$$

which is the payoff of a put option on S. The present value v_i of the loss at t_i can be expressed in terms of the Black-Scholes formula, and total expected loss is $\sum_{i=1}^{N} q_i v_i$.

6.9 Some Typical Credit Derivatives

- Credit Default Swaps (CDS)

  The most widely used type of credit derivatives is the credit default swap (CDS). It works as follows: it is a swap contract in which one party A pays a premium in terms of a rate to the other party (B) on a scheduled time table until the maturity T. In the event of a default of the specified reference entity, party A stops the premium payments and receives the incurred loss amount from party B.

  For a CDS, the issues involved are obviously pricing and hedging to begin with. The main parameter in the contract is the premium rate, which is more or less the yield spread of a bond with that maturity issued by this reference entity. This rate s is specified when the contract is entered and stays fixed for the rest of the swap life, and it is set up so that the initial value of the swap is zero. As market conditions change, the prevailing spread s would change so the value of the swap can be positive or negative. The valuation of the swap is similar to interest rate swaps, except we have to take into considerations of the default at all these times.

  The valuation of a CDS is quite simple. On one side that pays the premium, the present value of all the payments is

$$\sum_{i=1}^{n} V(t_i) s D(t_i),$$

here s is the premium rate, $D(t_i)$ is the discount factor for payments received at t_i, and $V(t_i)$ is the survival probability for t_i. The present value of the other side is

$$\sum_{i=1}^{n} q_i(1-R)D(t_i).$$

here we assume the principal \$1, with payments made annually, and a recovery rate R. The rate s that makes the contract value zero is

$$s = \frac{\sum_{i=1}^{n} q_i(1-R)D(t_i)}{\sum_{i=1}^{n} V(t_i)D(t_i)}$$

- Collaterized Default Obligations (CDO)

  These are the notorious structured notes in which a collection of defaultable bonds/notes are structured according to a detailed schedule, as to specify where to send the income generated by the bonds in terms of a set of tranches. The senior tranches get their first pick of the returns, and the junior tranches will only get the leftover. The trick is the use of some high quality bonds to cover the problematic assets in the portfolio, therefore to sell it as a high rating security. In the following example we will show how this simple trick can mislead the public into believing that the investment is a sound choice.

6.10 First-to-default (first-to-exit) models

In many CDS/CDO modeling, the first-to-default, or second-to-default issues would arise. Suppose $\tau_1, \tau_2, \ldots, \tau_m$ are default times of companies $1, 2, \ldots, m$, each with a survival probability

$$p_i = \mathbb{P}\left[\tau_i > t\right]$$

How do we determine the distribution of the first-to-default, or first-to-exit time $\tau = \min_i \{\tau_i\}$?
In the case all the company defaults are **independent** from each other,

$$\begin{aligned}
\mathbb{P}[\tau > t] &= \mathbb{P}\left[\tau_1 > t, \tau_2 > t, \ldots, \tau_m > t\right] \\
&= \mathbb{P}[\tau_1 > t] \cdot \mathbb{P}[\tau_1 > t] \cdots \mathbb{P}[\tau_m > t] \\
&= p_1 \cdot p_2 \cdots p_m
\end{aligned}$$

This simplifies the problem quite a bit, and it can explain the popularity with such intensity-based models. However, this convenience in practice effectively encourages many practitioners to assume this all-too-important, but not necessary realistic, **independence** assumption. This oversimplifyication could be devastating in many applications.

6.11 Copula Model

Now that we realized that the independence assumption is not a reasonable one, and the correlation factor is often a major issue in many structured products such as CDOs, it becomes clear to practitioners that the dominant issue is to model correlations. The copula model makes an often oversimplified attempt to address this issue. It is observed that the random variable τ can be transformed to another random variable with **uniform distribution**: Suppose $p(t) = \mathbb{P}[\tau_i > t]$ is the survival probability, the inverse CDF method suggests that if we take a uniformly distributed rv $U \sim$ Unif $[0,1]$, then $\tau = p^{-1}(U)$ has $1 - p(t)$ as its CDF. The idea of the copula model is that instead of working with rv's τ_i with individual distributions and a rather special correlation structure, it would be far simpler to work with transformed rv's U_i, for which the correlation structure may be much easier to specify. The copula model thus changes the problem of imposing a correlation structure for $\tau_1, \tau_2, \ldots, \tau_m$ into a correlation structure for $U_1, U_2, \ldots, U_m$. Namely, we call

$$C(u_1, u_2, \ldots, u_m) = \mathbb{P}(U_1 \leq u_1, \ldots, U_m \leq u_m),$$

a copula function for the transformed rv's $U_1, U_2, \ldots, U_m$. A particular copula model specifies the form of the copula function C. Consider two rv's, for examples,

1. Independence: $C(u,v) = uv$;
2. Perfect correlation: $C(u,v) = \min(u,v)$;
3. Gaussian: $C(u,v) = P(N(X) \leq u, N(Y) \leq v)$ where X, Y are standard joint normal random variables with correlation coefficient ρ, and $N(x)$ is the cumulative normal distribution function.

Gaussian copula model is one of the most popular copula models in which a pair of joint Gaussian rv's with correlation coefficient ρ is simulated (which is easy to do), and they are turned to a pair of uniform distributed U, V, and then further turned to a pair τ_1, τ_2. The problem, however, is that the correlation between X and Y is **not** the same as the correlation between τ_1 and τ_2.