Chapter 5

Interest-Rate Modeling and Derivative Pricing

5.1 Basic Fixed-Income Instruments

The original fixed income security can be viewed as a stream of payments at future times $t_1, \ldots, t_n$, with **fixed** amounts of payments $c_1, \ldots, c_n$. In a zero-interest world, there would have been no question about its value. However, with non-zero interest rates and especially with randomly fluctuating interest rates, these future payments, though with known amounts, will have randomly fluctuating present values, and the valuation becomes a crucial aspect in selecting instruments to be included in investor's portfolios.

Fixed-income products will have two sources of risk:

- interest rate risk;
- credit risk.

Interset-rate risks had been considered and studied ever since there were loans, and some common economic factors include general economic conditions, monetary policies, and international trades, etc.

Credit risks refer to those related to potential failures of governments or firms to make the promised payments. They will be considered in the next chapter, and we assume no such risk in the discussions in this chapter.

Unlike a particular stock, we observe quite a few indicators related to current interest rates, such as

- LIBOR rates;
- bond prices (including government and corporate bonds);
- mortgage rates and rates for other loans such as credit cards.

5.1.1 Zero-Coupon Bonds

First let us consider so-called zero-coupon bonds, those bonds that come with no coupons, just a final lump sum (usually the face value) paid at maturity T. We define the following

$$Z(t,T) = \text{time-}t \text{ value of \$1 paid at } T$$

which is necessarily less than \$1 if there is a positive interest rate. On the other hand, we expect a positive return when we invest $Z(t,T)$ at t and receive \$1 back at T, which implies a return rate

$$y(t,T) = -\frac{1}{T-t}\log Z(t,T)$$

or

$$Z(t,T) = e^{-y(t,T)(T-t)}$$

Here we assume a continuously compounding in calculating the interest amounts.

For example, in the following table, for each row we observe one of the two from the market:

Table 5.1: Observed Interest Information

T (in years)	$Z(0,T)$	$y(0,T)$
2	0.99	0.5%
5	0.932	1.4%
10	0.8187	2.0%

One immediate use of the zero-coupon bond prices is to price a stream of cash flows promised by a coupon bond. Suppose a bond provides amounts c_i at $t_i,\ i=1,\dots,n$ and we use vector notation $\mathbf{c} = [c_1, c_2, \dots, c_n],\ \mathbf{T} = [t_1, t_2, \dots, t_n]$ with $t_n = T$, then the total time-t present value of the stream of cash flows is

$$V(t;\mathbf{c},\mathbf{T}) = \sum_{i=1}^{n} c_i Z(t,t_i) = \sum_{i=1}^{n} c_i e^{-y(t,t_i)(t_i-t)}$$

For example, with the information in the above table, suppose a security pays \$10 at $t=2$, \$10 at $t=5$, and \$100 at $t=10$, the current price should be

$$\$10 \times 0.99 + \$10 \times 0.932 + \$100 \times 0.8187$$

On the other hand, we can use the current coupon-bond price to recover zero-coupon bond prices, a procedure called boot-strapping. For example, suppose we have V_i denote the present value of an annuity that pays c_k at $t_k,\quad k=1,\dots,i$, then we have

$$Z(t,t_1) = \frac{V_1(t)}{c_1}, \quad Z(t,t_i) = \frac{V_i(t) - V_{i-1}(t)}{c_i},\ i=2,\dots,n.$$

5.1.2 Forward Rates

Definition 1 *The forward rate $f(t, T, T')$ is the rate that would have been applied to the time period $[T, T']$, implied from the interest rate market observed at t.*

Here *implied* means that it is a rate that would lead to no arbitrage opportunities for investors. This can be obtained by considering the following strategy at time t:

- purchase A units of maturity T zero-coupon bond, this will cost $A \cdot Z(t, T)$;
- short sell $A \cdot \frac{Z(t,T)}{Z(t,T')}$ units of maturity T' zero-coupon bond ($T' > T$), this will generate a sale proceed $A \cdot Z(t, T)$.

So the net cost at time t is zero. Now look at the future cash flows: you will receive A at T and pay $A\frac{Z(t,T)}{Z(t,T')}$ at T', this implies a locked-in rate of return

$$\frac{1}{T' - T} \log \frac{Z(t, T)}{Z(t, T')} = f(t, T, T')$$

This is defined to be the forward rate for period $[T, T']$. Any contracts promising a future return different from this forward rate will cause an arbitrage, this is why we justify this forward rate as implied from a no-arbitrage condition.

We can derive the relations among these three variables: $y(t, T)$, $f(t, T, T')$ and $Z(t, T)$. First we derive the forward rate from the yield y:

$$\begin{aligned} f(t, T, T') &= \frac{1}{T' - T} \left[-y(t, T)(T - t) + y(t, T')(T' - t)\right] \\ &= y(t, T')\frac{T' - t}{T' - T} - y(t, T)\frac{T - t}{T' - T}. \end{aligned}$$

As $t \to T$, we have $f(T, T, T') = y(T, T')$. As $T \to T'$, we have

$$f(t, T) = \lim_{T' \to T} f(t, T, T') = -\frac{\partial}{\partial T} \log Z(t, T) \tag{5.1}$$

To obtain Z from f, we can integrate the above equation

$$Z(t, T) = \exp\left(-\int_t^T f(t, u)\, du\right) \tag{5.2}$$

As $t \to T \to T'$, $y(t, t) = r(t)$ is called the short rate, which resembles the overnight rate that is applied for one day and it resets everyday.

Stochastic models for interest rates can be categorized into

1. short rate models: $dr(t) = a\, dt + b\, dW(t)$
2. HJM-forward rate models: $df(t, T) = a\, dt + b\, dW(t)$

Here a and b can take various forms with dependence on r or f themselves.

5.1.3 Arbitrage-Free Pricing of Bonds

We consider only short rate models in this discussion so the driving force is always $r(t)$, and we can view $Z(t,T)$ as the market response to the short rate change for various T maturities. One basic instrument that is affected by r in the most transparent way is the money market account and we introduce the money market price $B(t)$, with

$$B(0) = 1, \quad dB(t) = r(t)B(t)\,dt$$

and the solution is

$$B(t) = e^{\int_0^t r(s)\,ds}$$

There is nothing complicated with this formula if r is a known function of t. B as a function of t describes the change of the money market account and it is always growing as long as $r > 0$.

Now we can look at the opposite, the discount factor

$$D(t,T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r(s)\,ds}$$

which describes the discounting from t to T. Notice that unlike $Z(t,T)$, the discount factor $D(t,T)$ is not observed at t, and we usually work with the expectation of the discount factor in the stochastic interest rate models. One special case with the discount factor is when $t = 0$ and we introduce

$$D(T) = D(0,T) = \frac{1}{B(T)}$$

Theorem 1 *(Arbitrage free condition) The market is arbitrage-free if there exists a probability measure $\tilde{\mathbb{P}}$ such that the discounted zero-coupon bond price $\bar{Z}(t,T) = D(t)Z(t,T)$ is a martingale for each T, that is*

$$\bar{Z}(t,T) = \tilde{\mathbb{E}}_t\left[\bar{Z}(T,T)\right] = \tilde{\mathbb{E}}_t\left[D(T)\right]$$

Once we have this probability measure, we can express

$$\begin{aligned}
Z(t,T) &= \frac{1}{D(t)}\bar{Z}(t,T) \\
&= \tilde{\mathbb{E}}_t\left[B(t)D(T)\right] \\
&= \tilde{\mathbb{E}}_t\left[\frac{B(t)}{B(T)}\right] && (5.3) \\
&= \tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)\,ds}\right] && (5.4)
\end{aligned}$$

If we have a derivative that has a payoff $V(T)$ at T, the value of the derivative at $t < T$ should be

$$V(t) = \tilde{\mathbb{E}}_t\left[\frac{B(t)}{B(T)}V(T)\right] = \tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)\,ds}V(T)\right]$$

For example, if we have a forward contract on X, with payoff $X(T) - K$, the time t price can be computed according to

$$V(t) = \tilde{\mathbb{E}}_t\left[\frac{B(t)}{B(T)}(X(T) - K)\right] = X(t) - KZ(t,T)$$

5.2 Single-Factor Models

Even though there are so many rates changing in time as variables, the first step in modeling is to assume that there is a single factor driving all the rates. The short rate $r(t)$ is usually designated to serve this role. Together with the no-arbitrage assumption, the short rate can imply the other rates in a single factor setting. Another motivation is that from Eq.(5.4) we can calculate the zero-coupon bond price at t, and once we have $Z(t,T)$, we can calculate the (instantaneous) forward rate

$$f(t,T) = -\frac{\partial}{\partial T}\log Z(t,T)$$

How do we calculate the zero-coupon bond price $\tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)\,ds}\right]$? We look at following examples:

1. $r = const$

$$Z(t,T) = \tilde{\mathbb{E}}_t\left[e^{-r(T-t)}\right] = e^{-r(T-t)}$$

2. $dr(t) = \sigma\,dW(t)$

   We can solve for r as

$$r(t) = r_0 + \sigma W(t)$$

   so

$$\begin{aligned} Z(t,T) &= \tilde{\mathbb{E}}_t\left[e^{-r_0(T-t)-\sigma\int_t^T W(s)\,ds}\right] \\ &= e^{-r_0(T-t)}\cdot\tilde{\mathbb{E}}_t\left[e^{-\sigma X}\right] \end{aligned}$$

   where $X = \int_t^T W(s)\,ds$ is a normal random variable with mean 0 and a variance calculated as below. To find the variance, we use integration by parts (Itô's formula version) to express

$$\begin{aligned} X &= \int_t^T W(s)\,ds \\ &= TW(T) - tW(t) - \int_t^T s\,dW(s) \\ &= T(W(T) - W(t)) + (T-t)W(t) - \int_t^T s\,dW(s) \\ &= \int_t^T (T-s)\,dW(s) + (T-t)W(t) \end{aligned}$$

We notice that the integral in the above involves Brownian increments from t to T, which are independent of $W(t)$, so the variance is

$$\text{Var}[X] = \int_t^T (T-s)^2\, ds + (T-t)^2 t = \frac{1}{3}(T-t)^2(T+2t)$$

With the variance of X calculated, we have the zero-coupon bond price

$$Z(t,T) = e^{-r_0(T-t)+\frac{\sigma^2}{3}(T-t)^2(T+2t)}$$

We can see that even in such a simple process the calculations of Z can be quite involving. Are there alternatives, and hopefully simpler ways to calculate $Z(t,T)$? It turns out that the PDE approach can help.

As $\bar{Z}(t,T)$ is a martingale, we have

$$d\bar{Z}(t,T) = \theta(t)d\tilde{W}(t)$$

for some adapted process $\theta(t)$. On the other hand,

$$\begin{aligned} d\bar{Z}(t,T) &= d\left(D(t)Z(t,T)\right) \\ &= D(t)\, dZ(t,T) - rD(t)Z(t,T)\, dt \end{aligned}$$

Setting two expressions equal, we have

$$\frac{dZ}{Z} = r\, dt + \frac{\theta(t)}{D(t)Z} d\tilde{W} = r\, dt + \sigma_Z\, d\tilde{W}$$

If we use a short rate model, using the Markov property,

$$\begin{aligned} Z(t,T) &= \tilde{\mathbb{E}}_t\left[e^{-\int_t^T r(s)\, ds}\right] \\ &= \tilde{\mathbb{E}}\left[e^{-\int_t^T r(s)\, ds} | r(t) = r\right] \\ &= Z(t,T,r) \end{aligned}$$

Compare this with the stock derivative pricing formula

$$V = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\Lambda(S(T))|S(t) = S\right] = V(t,S)$$

the zero-coupon bond price $Z(t,T,r)$ should satisfy a similar PDE in variables t and r. To derive the PDE, we assume the short rate r follows

$$dr(t) = (a - \gamma b)\, dt + b\, d\tilde{W}$$

where a and b can both depend on t and r, and γ is a given parameter. Following the derivation of Black-Scholes-Merton PDE, the equation satisfied by $Z(t,T,r)$ for a fixed T is

$$\frac{\partial Z}{\partial t} + \frac{b^2}{2}\frac{\partial^2 Z}{\partial r^2} + (a - \gamma b)\frac{\partial Z}{\partial r} - rZ = 0 \tag{5.5}$$

with a terminal condition $Z(T, T, r) = 1$.

How do we solve such a PDE problem? For a type of models (called affine models) we may suggest a separation of variables approach

$$Z = e^{A(t,T) - C(t,T)r},$$

which basically assumes that $\log Z$ is linear in r (therefore the name "affine"). Consider the case $\gamma = 0,\ a = a_0 + a_1 r,\ b^2 = b_0 + b_1 r$, after substitute the assumed solution form, we end up with two ODEs for A and C:

$$\frac{dA}{dt} - a_0 C + \frac{b_0}{2} C^2 = 0 \tag{5.6}$$

$$\frac{dC}{dt} + a_1 C - \frac{b_1}{2} C^2 + 1 = 0 \tag{5.7}$$

Example: Consider Vasicek model

$$dr = (\alpha - \beta r)\, dt + \sigma\, dW \tag{5.8}$$

in which $a_0 = \alpha,\ a_1 = -\beta\ b_0 = \sigma^2,\ b_1 = 0$, the ODEs to be solved are

$$\frac{dA}{dt} - \alpha C + \frac{\sigma^2}{2} C^2 = 0 \tag{5.9}$$

$$\frac{dC}{dt} - \beta C + 1 = 0 \tag{5.10}$$

with terminal conditions $A(T, T) = C(T, T) = 0$. The solutions for this system of ODEs are quite easy to obtain:

$$C = \frac{1 - e^{-\beta(T-t)}}{\beta}, \tag{5.11}$$

and

$$A = -\int_t^T \left(\alpha C - \frac{\sigma^2}{2} C^2 \right) ds. \tag{5.12}$$

The Vasicek model is quite popular in that it includes a mean-reversion part and the explicit solution is quite simple. The disadvantage is that it allows the short rate to be negative, which is unsettling matter for many practitioners.

Other popular models include

- Cox-Ingersoll-Ross (CIR) model:

  $$dr = (\alpha - \beta r)dt + \sigma \sqrt{r}\, dW$$

  This model is one of the square-root processes and it guarantees $r > 0$ almost surely with the Feller condition satisfied.

- Ho-Lee model:
$$dr = \alpha(t)\, dt + \sigma\, dW$$
This allows the whole yield curve to be fitted with a time-dependent drift term α.

- Black-Derman-Toy model
$$dr = \alpha(t) r(t)\, dt + \sigma(t)\, dW$$
This allows both the drift and volatility to be time-dependent, and it is natural for a binomial tree implementation.

5.3 Fixed Income Derivatives

Fixed income derivatives are those financial instruments with values determined by some interest rates. In the following we discuss a few examples.

5.3.1 Swaps, Caps/Floors, and Swaptions

A. Interest rate swaps

These are the simplest type of interest rate derivatives, and they have been in use extensively and are very well understood. The purpose of an interest rate swap is to be locked into a fixed rate payment for a future period of time. There are two parties involved and they both have loans with the same principal amount and same maturity, but one pays a fixed interest rate (k) and the other pays a floating interest ($f(t)$ at t). An interest rate swap is an agreement between these two parties to exchange their interest payments (to be made every pay period, such as a month, a quarter, or a year). One example is a 10-year swap in which party A pays party B a fixed annual 3% interest rate, and receives a floating rate (such as the 6-month LIBOR rate). The interest payments are exchanged every 6 month. Suppose the principal is \$1 million, and the current LIBOR is 1%, then at $t = 0.5$ (6 months), party A pays party B the amount $\$1,000,000 \times (3\% - 1\%) \times 0.5 = \$10,000$. At $t = 0.5$, suppose the ongoing LIBOR goes to 1.5%, then party A pays B an amount \$7,500. Of course the value of the contract is going to be determined by the prevailing interest rate market, more specifically, the current yield curve. There is a particular fixed rate that would make the contract "fair" to both parties and this rate is called the T-maturity swap rate, which fluctuates in time and it is a very important market indicator.

B. Interest rate caps/floors

An interest rate cap contract is a collection of individual caplets, each one is a call option on the specified interest rate at a specified time. For example, a caplet for the period $[T, T+\tau]$ provides a payoff $\Lambda = \tau\, (f(T) - k)^+$ at time $T + \tau$. The motivation of an interest rate cap is the following: an investor has an outstanding loan in which the interest payments are calculated according to a floating rate $f(t)$ for the period $[t, t + \tau]$. The investor is concerned about the potential rise of the interest rate and he/she is only prepared to pay a rate that is no higher than k. If this is the case the investor can invest in an interest cap, then at any time t if the interest rate $f(t)$ is higher than k, a caplet will kick in to pay the difference between $f(t)$ and k. In the end, the borrower is guaranteed to pay a rate that is no higher than k.

The pricing of an interest cap can be split into pricing of individual caplets, with each one based on a formula that is analogous to the Black-Scholes formula.

Interest rate floors to interest rate caps are just like call options to put options. They guarantee a minimum interest payment to the lender and the pricing can be derived from the pricing of caps.

C. Options on bonds

For simplicity, we use a zero-coupon bond with maturity T' for example. The price of the bond at time t is $Z(t, T)$, and suppose that the investor thinks the bond price will go up at time $0 < T < T'$, he can enter a call option on the bond with a payoff at T:

$$\Lambda = (Z(T, T') - K)^+$$

here K is specified in the contract as the strike price. The time-t value of the call option in our framework can be established as

$$V(t) = \tilde{\mathbb{E}}_t \left[D(t, T) \left(Z(T, T') - K \right)^+ \right]$$

In the following we present the basic formulations in pricing these three derivatives.

5.3.2 Pricing of Interest Rate Swaps

An interest rate swap may begin with value zero when it was set up. As market conditions change, the value will fluctuate and the pricing of swap contracts is quite straightforward. We suppose a swap has the following parameters:

- maturity T,
- time period related to payment frequency τ,
- fixed rate k,
- floating rate $f(t)$.

The pricing of the swap is based on the total PVs of both sides (legs). Assuming \$1 principal, the floating side payments should have total present value \$1 (valued at par), since the floating interest rate payments adjust automatically to the market and they serve as the reference for the market. Suppose f_i is the floating rate applied to period $[T_{i-1}, T_i]$ and it is observed at T_{i-1}, the above statement is

$$\tilde{\mathbb{E}}\left[\sum_{i=1}^{n} D(0,T_i) f_i \tau + D(0,T)\right] = 1$$

The fixed legs have total present value

$$\tilde{\mathbb{E}}\left[\sum_{i=1}^{n} D(0,T_i) k\tau + D(0,T)\right] = \sum_{i=1}^{n} Z(0,T_i) k\tau + Z(0,T)$$

The value of the swap to party A (paying fixed) is therefore

$$V(0) = 1 - \sum_{i=1}^{n} Z(0,T_i) k\tau - Z(0,T) \tag{5.13}$$

The swap rate for maturity $T = T_n$, that is the fixed rate that would make the swap valued zero to both parties, is

$$k = \frac{1 - Z(0,T)}{\tau \sum_{i=1}^{n} Z(0,T_i)} \tag{5.14}$$

The extension of Eq.(5.15) to $t = T_j$ is

$$V(t) = 1 - \sum_{i=j+1}^{n} Z(t,T_i) k\tau - Z(t,T) \tag{5.15}$$

If you want to price the swap at some time not exactly on one of the payment dates, certain adjustments need to be made.

5.3.3 Pricing of Caps/Floors

As we mentioned, an interest rate cap is a collection of interest rate caplets, so for pricing we should just focus on an interest rate caplet. Consider one such caplet in which a payoff in the amount $(f-k)^+\tau$ occurs at time $T+\tau$ and the floating rate f is observed at T. The time-t value of this caplet is

$$\text{Caplet}_{T+\tau}(t) = \tau \tilde{\mathbb{E}}_t\left[D(t, T+\tau)(f-k)^+\right]$$

With the notation $\mathbf{T}$ to include all payment dates, the price of the cap is

$$\text{Cap}(t;\mathbf{T}) = \sum_{i=1}^{n} \text{Caplet}_{T_i}(t) = \tau \tilde{\mathbb{E}}_t\left[\sum_{i=1}^{n} D(t,T_i)(f_i - k)^+\right]$$

Similarly a floor contains a collection of floorlets and the time-t price is

$$\text{Floor}(t; \mathbf{T}) = \tau \tilde{\mathbb{E}}_t \left[\sum_{i=1}^{n} D(t, T_i)(k - f_i)^+ \right]$$

The fact that the payments are made at T_i while the rate is observed at T_{i-1} allows an interpretation that is helpful in pricing caplets. To see this, we notice that the time T_{i-1}-value of this cash flow at T_i should be

$$(f_i - k)^+ \tau Z(T_{i-1}, T_i)$$

On the other hand, f_i is the rate revealed at T_{i-1} that is supposed to cover time period $[T_{i-1}, T_i]$, assuming simple compounding, we have

$$Z(T_{i-1}, T_i) = \frac{1}{1 + f_i \tau}$$

With this observation, the time T_{i-1}-value of the cash flow at T_i

$$\begin{aligned}
&\tau(f_i - k)^+ Z(T_{i-1}, T_i) \\
=&\tau \left(\frac{f_i}{1 + f_i \tau} - \frac{k}{1 + f_i \tau} \right)^+ \\
=&\tau \left(1 - \frac{1 + k\tau}{1 + f_i \tau} \right)^+ \\
=&\tau(1 + k\tau) \cdot \left(\frac{1}{1 + k\tau} - Z(T_{i-1}, T_i) \right)^+
\end{aligned}$$

which is $\tau(1+k\tau)$ times a payoff of a put option on Z with strike $1/(1+k\tau)$. So the caplet can be priced as a put option on $Z(T_{i-1}, T_i)$, which is observable from the market. If we can estimate the volatility of this bond price, we should be able to use a Black-Scholes type formula to price caplets. However, we should note that volatilities used in pricing caplets are different from one caplet to another. In reality, an average volatility is used in such a formula.

5.3.4 Pricing of Swap Options/Bond Options

Finally, we discuss a few points regarding pricing swap options (swaptions) and bond options. Zero-coupon bond options are relatively simple to valuate in short rate models, and one approach is to use the PDE in Eq.(5.5) for the option price V. Suppose we want to price a call with expiration T, on the zero-coupon bond that matures at $T' > T$, the value of the option at time t when the observed short rate is r satisfies the PDE

$$\frac{\partial V}{\partial t} + \frac{b^2}{2} \frac{\partial^2 V}{\partial r^2} + (a - \gamma b) \frac{\partial V}{\partial r} - rV = 0 \tag{5.16}$$

with the terminal condition

$$V(T, r) = \max\left(Z(T, T', r) - K, 0\right)$$

where $Z(T, T', r)$ is the solution of Eq.(5.5) with terminal condition $Z(T', T', r) = 1$. Obviously we do not expect a closed-form solution, but many numerical solutions can be obtained using various numerical methods.

The other methods will require a so-called forward measure to price. In our previous attempts, we assume the numeraire to be the money market account

$$B(t) = B(0)e^{\int_0^t r(s)\,ds}$$

For bond options, it is more convenient to use a different numeraire and this time the zero coupon bond seems to be a better choice. The complication is that we have different maturities for different zero-coupon bonds so in choose a numeraire we need to specify the particular T maturity bond. If we specify a maturity T, the T-forward measure will lead to $V(t)/Z(t, T)$ to be a $\hat{\mathbb{P}}$-martingale.

Finally, we want to mention that swap options can be made equivalent to bond options. Suppose we have an option to enter a swap that receives a fixed rate k and pays the floating rate, it is the same as an option to purchase a bond that pays coupon rate k, as the floating legs are assumed to be valued at par. Therefore, most swaption pricing methods can be obtained by modifying from bond option pricing.