# Chapter 4

# Risk-Neutral Pricing of Derivatives in the (B, S) Economy

## 4.1 (B,S) Economy

We have two *tradable* assets in the (B,S) economy: (1) a bond (B) with a guaranteed (risk-less) growth with annualized rate $r$, and a stock (S) with uncertain (risky) growth, and their dynamics in the **risk-neutral** world are described as follows.

- Bond:

$$\begin{aligned} &\text{model:} \quad dB = rB\,dt \\ &\text{solution:} \quad B(t) = B(0)e^{rt} \end{aligned}$$

- Stock:

$$\begin{aligned} &\text{model:} \quad \frac{dS}{S} = r\,dt + \sigma\,d\tilde{W} \\ &\text{solution:} \quad S(t) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)+\sigma\tilde{W}(t)} \end{aligned}$$

A **derivative, or contingent claim** on $S$ with payoff $V_T = \Lambda(S)$ at $T$ is a contract where at the expiration $T$ one party (seller of the contract) pays the other party (buyer of the contract) an amount $V_T$ which is determined from $S(T)$ (or $S(t), 0 \le t \le T$) in a predetermined formula $\Lambda(S)$. The seller receives a compensation by charging the buyer a premium at the onset of the contract. One major advantage of such contracts is that these contracts can be bought and sold on the market after the signing any time before the expiration.

The main question is: what is the no-arbitrage price (the premium that will not lead to any arbitrage opportunities for investors) of this contingent claim, at time $t < T$? This is obviously the question that every market participant will be asking during the life of this derivative product.

We anticipate that the price at $t$ should be determined based on all the information leading to time $t$, hopefully just the information at time $t$. In another word, the price should not depend on information not yet revealed to the market.

## 4.2 No-arbitrage Price

The idea to determine this no-arbitrage price is the following. We try to construct a *replicating* portfolio consisting the bond and the stock, that will be "equivalent" to the derivative payoff at $T$, or that it does the same job as the derivative. If we can find such a portfolio, then the price of the derivative should be exactly the same as the value of the portfolio, otherwise there will be arbitrage opportunities. While the value of the portfolio can be determined because the prices of the stock and the bond are readily observed at any time of valuation. But the looming question is how many shares of the stock, and how many units of the bond we shall include in the portfolio to do that "replicating" job.

In a (B,S) economy, any portfolio consisting of these two assets can be expressed as

$$\Pi_t = \Delta_t \cdot S(t) + \beta_t \cdot B(t)$$

where $\Delta_t$ is the number of the shares of the stock at time $t$, and $\beta_t$ is the number of units of the bond at time $t$. We can denote a portfolio just by $(\Delta_t, \beta_t)$.

For a portfolio that replicates the derivative with payoff $V_T$, we must have

1. $\Pi_T = V_T$ at $T$, in **every** possible scenario,

2. the portfolio is self-financing.

Here self-financing means that there will be no fund taken out, and no fund injected during the trading period $0 < t < T$. In mathematical terms, it amounts to the condition

$$S(t)\, d\Delta_t + B(t)\, d\beta_t = 0$$

Therefore the change of portfolio value will be

$$d\Pi_t = \Delta_t\, dS(t) + \beta_t\, dB(t)$$

If a portfolio is self-financing, we can express the value as

$$\Pi_t = \Pi_0 + \int_0^t \beta_u\, dB(u) + \int_0^t \Delta_u\, dS(u)$$

We shall use the models for $B(t)$ and $S(t)$, with the help of Itô calculus, to derive the differential of the discounted portfolio value $\bar{\Pi}_t = e^{-rt}\Pi_t$,

$$\begin{aligned}
d\bar{\Pi}_t &= -re^{-rt}\,\Pi_t\, dt + e^{-rt}\left(\beta_t\, dB(t) + \Delta_t\, dS(t)\right) \\
&= -re^{-rt}\,\Pi_t\, dt + e^{-rt}\left[r\left(\Pi_t - \Delta_t S(t)\right)\, dt + \Delta_t\, dS(t)\right] \\
&= e^{-rt}\Delta_t\left(-rS(t)\, dt + dS(t)\right) \\
&= e^{-rt}\sigma S(t)\, d\tilde{W}
\end{aligned}$$

This shows that $\bar{\Pi}_t$ is a martingale under the risk-neutral probability measure, for **any** self-financing portfolios. In particular, with a martingale we can express

$$\bar{\Pi}_t = \tilde{\mathbb{E}}_t \left[\bar{\Pi}_T\right]$$

Let us first pretend that it is indeed possible to replicate the derivative payoff, that is, to have a self-financing portfolio $\Pi_t$ such that

$$\Pi_T = V_T = \Lambda(S(T))$$

then we have

$$\bar{\Pi}_t = \tilde{\mathbb{E}}_t \left[e^{-rT}\Lambda(S(T))\right]$$

If we are successful in creating this replicating portfolio, the value of the portfolio should be the same as the price of the derivative. Now that we have the no-arbitrage price

$$\boxed{V_t = \Pi_t = e^{-r(T-t)}\tilde{\mathbb{E}}_t \left[\Lambda(S(T))\right]}$$

once we have the conditional distribution of $S(T)$, given $S(t)$, then it's just a matter of calculating the expectation. This is theoretically straightforward, but we can rarely get a closed-form solution. There is, however, another approach to solve the same problem: the partial differential equation approach. First we note that

$$\bar{V}_t = \tilde{\mathbb{E}}_t \left[\bar{V}_T | S(t) = S\right] = \bar{u}(t, S) = e^{-rt}u(t, S)$$

is a function of $t$ and $S$. We can use Itô's formula to calculate

$$\begin{aligned} d\bar{u} &= -re^{-rt}u\,dt + e^{-rt}\,du \\ &= e^{-rt}\left[\left(u_t + rSu_S + \frac{1}{2}\sigma^2 S^2\,u_{SS} - ru\right)dt + \sigma S u_S\,d\tilde{W}\right] \end{aligned}$$

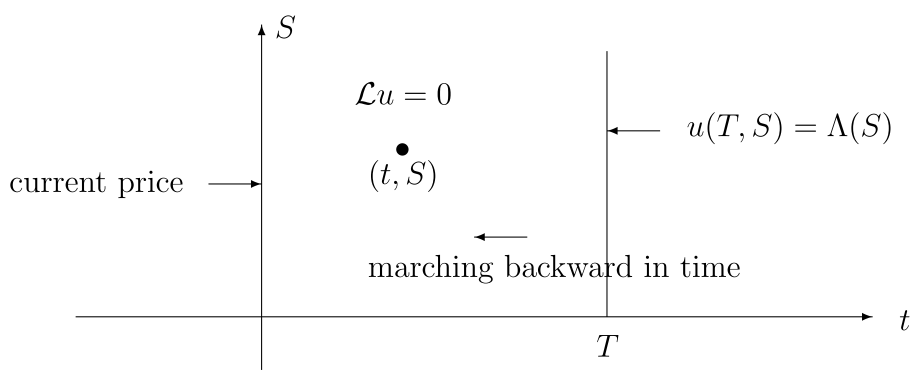
If the term in the bracket

$$u_t + rSu_S + \frac{1}{2}\sigma^2 S^2\,u_{SS} - ru = 0$$

then $\bar{u}$ is a martingale!

$$\bar{u}(t, S(t)) = \tilde{\mathbb{E}}_t \left[\Lambda(S(T))e^{-rT}\right]$$

This says that if $u$ satisfies this Black-Scholes-Merton partial differential equation, then $\bar{u}$ is just the conditional expectation we have been trying to compute. So the problem is turned to solving the PDE problem for $u(t, S)$ illustrated in the following diagram.



**Example 1** *The Black-Scholes formula for call and put options can be viewed as solution to the following PDE problem:*

$$u_t + rSu_S + \frac{1}{2}\sigma^2 S^2 u_{SS} = ru$$

*with terminal condition*

$$u(T, S) = \max(S - K, 0)$$

*for call options or*

$$u(T, S) = \max(K - S, 0)$$

*for put options.*

## 4.3 Justification of the no-arbitrage price

Now is the final piece for the puzzle: how do we make sure that $\Pi_T = V_T$ for all scenarios? We can just make sure that they start the same value, and all the increments in time also match in all scenarios.

1. Starting at the same value:
$$V_0 = \Pi_0$$

2. Matching increments:
$$d\left[\bar{V}(t, S(t))\right] = d\bar{\Pi}_t$$

The change of $\bar{V}$ is

$$d\left[e^{-rt}V(t, S(t))\right] = e^{-rt}\sigma S\frac{\partial V}{\partial S}\,d\tilde{W}$$

and from our earlier calculation,

$$d\bar{\Pi}_t = \Delta_t\sigma\bar{S}(t)\,d\tilde{W} = e^{-rt}\sigma S\Delta_t\,d\tilde{W}$$

In order to have $d\bar{\Pi}_t = d\bar{V}$, we can take

$$\Delta_t = \frac{\partial V}{\partial S}$$

Now we have a complete argument to justify this no-arbitrage price:

1. At time 0, we compute $V_0 = \tilde{\mathbb{E}}\left[e^{-rT}V_T(S(T))\right]$ from the expectation or by solving the PDE problem;

2. Invest $\Pi_0 = V_0$ in a portfolio that consists of $\Delta_0$ shares of $S$, and the rest in bonds;

3. At any time $t > 0$, rebalance the portfolio according to
$$\Delta_t = \frac{\partial V}{\partial S}|_t, \quad \beta_t = \frac{\Pi_t - \Delta_t S(t)}{B(t)}$$

4. At $T$, $\Pi_T = \Lambda(S(T))$, the portfolio will replicate the derivative payoff.

**Claim**: $\Pi_0$ is the no-arbitrage price of the derivative as it is the cost to replicate it.

## 4.4 Hedging of a derivative

There is yet another approach to derive the same no-arbitrage price. Instead of constructing a replicating portfolio, we consider the following situation: suppose you are the seller of the derivative product, how should you prepare for the outcomes? Or in a trading language, how should you **hedge** this short position? If it's a call option, your intuition tells you that you should buy some stock, but exactly how many shares? For this question, let us consider another portfolio with total value $P_t$;

$$P_t = V_t - \Delta_t \cdot S(t)$$

If you compute the change in time,

$$\begin{aligned} dP_t &= dV_t - \Delta_t \, dS \\ &= \left( u_t + rSu_S + \frac{1}{2}\sigma^2 S^2 \, u_{SS} \right) dt + \sigma S u_S \, d\tilde{W} - \Delta_t \left( rS \, dt + \sigma S \, d\tilde{W} \right) \\ &= \left( u_t + rS(u_S - \Delta_t) + \frac{1}{2}\sigma^2 S^2 \, u_{SS} \right) dt + \sigma S(u_S - \Delta_t) \, d\tilde{W} \end{aligned}$$

and we hope that the risk can be eliminated. This can be achieved by

$$\Delta_t = \frac{\partial V}{\partial S} = u_S$$

But then this portfolio is risk-less, it **must** make a risk-less return (with rate $r$). So we must have

$$dP_t = rP_t \, dt$$

or

$$u_t + rS(u_S - \Delta_t) + \frac{1}{2}\sigma^2 S^2 \, u_{SS} = r(u - \Delta_t S) = ru - rSu_S$$

which leads to the same Black-Scholes-Merton PDE for $u(t, S)$ as above.