# Chapter 2

# Brownian Motion (continued)

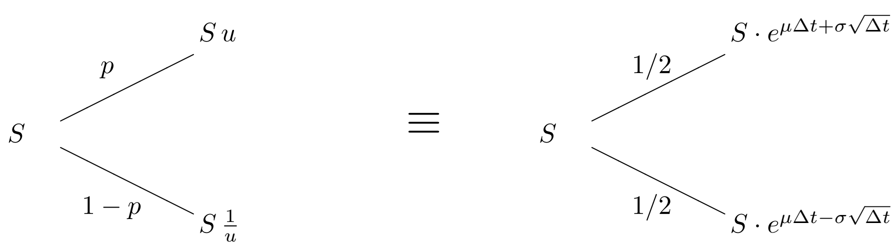
## 2.4 Using Brownian Motion to Model Stock Prices

Now we are ready to build a continuous time stock price model that is based on Brownian motion, as the limiting case of the stock price in the binomial model as the number of nodes goes to infinity. First we choose a time horizon $T > 0$, and divide it into $N$ equal subintervals, and we denote $t = n\Delta t$ so

$$S_n = S_0 u^{M_n} \approx S(t) = S(n\Delta t), \quad 0 \le n \le N$$

so we can connect the price in the binomial model to the price in the continuous time model.

First of all, we rather prefer to use symmetric random walk as shown in our limiting procedure for Brownian motion. It turns out that we can modify the binomial tree to achieve this:




Here we take $u$ to be $e^{\sigma\Delta t}$ and multiply a factor $e^{\mu\Delta t}$ to compensate the change the probability from $\tilde{p}$ to $1/2$ to lead to symmetric random walk. The equivalence means that the first two moments are matched, thus two equations are provided to solve for $\mu$ and $\sigma$. The factor $\sqrt{\Delta t}$ involved is now obvious from the construction of the scaled random walk. The stock price from this binomial model can be summarized as

$$S_n = S_0 e^{\mu n\Delta t+\sigma\sqrt{\Delta t}M_n} = S_0 e^{\mu t+\sigma\sqrt{t/n}M_n}$$

where $M_n$ is now a symmetric random walk. We will let $n \to \infty$ and $\Delta t \to 0$ such that $n\Delta t = t$, then as $n \to \infty$

$$\mu t + \sigma \sqrt{t/n} M_n = \mu t + \sigma \frac{1}{\sqrt{n'}} M_{\lfloor n' t \rfloor} \Longrightarrow \mu t + \sigma W(t)$$

here $n' = n/t \to \infty$ as $n \to \infty$. Since we expect $S_n \to S(n\Delta t) = S(t)$, we now have a continuous time stock price model

$$S(t) = S(0) e^{\mu t + \sigma W(t)}$$

and it is driven by the standard Brownian motion $W(t)$. The stock price in this model follows a *geometric Brownian motion*, since

$$\log S(t) = \log S(0) + \mu t + \sigma W(t)$$

is a Brownian motion with a drift term $\mu t$, and $\mu$ gives the slope that represents the trend.

In the world of positive interest rate, we are interested in the *discounted stock price*

$$\tilde{S}(t) = e^{-rt} S(t).$$

Using the previous exercise we can see that it is a martingale if

$$\mu - r = -\frac{1}{2}\sigma^2$$

or

$$S(t) = S(0) e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

Notice that the only parameters matter here are the risk-free interest rate $r$ and the volatility $\sigma$.

In general, we can categorize this model as one special case of the models

$$S(t) = S(0) e^{X(t)}$$

and we describe $X(t)$ in its differential

$$dX(t) = \mu \, dt + \sigma \, dW(t)$$

where we say that the infinitesimal increment

$$dW(t) = W(t + dt) - W(t) \sim N(0, dt).$$

This will lead us to an important type of processes called Itô's process:

$$dX(t) = \mu(X, t) \, dt + \sigma(X, t) \, dW(t)$$

# Chapter 3

# Continuous-time Stochastic Calculus

Suppose $X(t)$ is a continuous-time process as just described, with uncertainties coming from a Brownian motion, and we have another process defined by

$$Y(t) = f(X(t), t)$$

where $f(x,t)$ is a differentiable function of $x$ and $t$. How do we describe $Y(t)$? As $X(t)$ is described by these increments, we are naturally interested in $dY(t)$, which leads to a differential equation description of $Y$, and in this case we are dealing with stochastic differential equations (SDE). To study differential equations, we must be able to define differentials and here we need a formula for $dY$ in terms of $dX$. The answer is the famous Itô's formula

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2 \\ &= \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}\,dW \end{aligned}$$

You may wonder why we added the second-order term in the Taylor expansion. The reason is that we cannot ignore $(dW)^2$ as

$$\mathbb{E}[(dW)^2] = dt,$$

so we need to include terms like $(dX)^2$ in the expansion. Suppose we move from $t$ to $t + \Delta t$, denote $\Delta W = W(t+\Delta t) - W(t)$, $\Delta X = X(t+\Delta t) - X(t)$, then we can use Taylor's expansion to obtain

$$\begin{aligned} \Delta Y &= Y(t+\Delta t) - Y(t) \\ &= f(X(t+\Delta t), t+\Delta t) - f(X(t), t) \\ &\approx f_t \cdot \Delta t + f_x \cdot \Delta X + \frac{1}{2} f_{xx} \cdot (\Delta X)^2 \\ &\approx f_t \cdot \Delta t + f_x \cdot (\mu\Delta t + \sigma\Delta W) + \frac{1}{2} f_{xx} \cdot (\mu\Delta t + \sigma\Delta W)^2 \end{aligned}$$

For the last term, we have

$$(\mu\Delta t + \sigma\Delta W)^2 = \mu^2(\Delta t)^2 + 2\mu\sigma \cdot \Delta t \cdot \Delta W + \sigma^2(\Delta W)^2$$

and we can estimate these terms one by one:

- $\mu^2(\Delta t)^2$ is a higher order term (compared to $\Delta t$), so it can be ignored.
- The term $\Delta t \cdot \Delta W$ is random, its expectation is 0 and the variance is $(\Delta t)^3$, or the standard deviation is $(\Delta t)^{3/2}$, so it should also be ignored.
- Finally we have the term proportional to $(\Delta W)^2$. Since $\Delta W \sim N(0, \Delta t)$, the expectation of $(\Delta W)^2$ is $\Delta t$ and the variance is $2(\Delta t)^2$. So we can just replace $(\Delta W)^2$ by $\Delta t$ in the expansion.

Based on the above argument, we should just retain only one term from the above square term, that is we can write

$$(\mu\Delta t + \sigma\Delta W)^2 \approx \sigma^2\Delta t$$

in the Taylor expansion. Therefore

$$\Delta Y \approx \left(f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}\right)\Delta t + \sigma f_x \Delta W$$

By taking $\Delta t \to 0$, we have the Itô's formula.

Corresponding to stochastic differential equation for $X$, there is an integral version:

$$X(t) = x_0 + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s)$$

The last term requires a new definition as it is the limit in the following form:

$$I(t) = \int_0^t \sigma(s)dW(s) = \lim_{n\to\infty}\sum_{i=1}^n \sigma(t_{i-1})\left(W(t_i) - W(t_{i-1})\right)$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, provided

$$\mathbb{E}\left[\int_0^T X^2\,dt\right] < \infty$$

and $\sigma(t)$ is revealed by the time $t$. This is called the Itô's integral and the following properties are often used in financial calculus:

$$\begin{aligned}
&\mathbb{E}\left[I(t)\right] = 0\\
&\mathbb{E}_s\left[I(t)\right] = I(s)\\
&\mathrm{Var}\left[I(t)\right] = \int_0^t \mathbb{E}\left[X^2(s)\right]\,ds
\end{aligned}$$

**Example 1** *Let* $f(x,t) = \log x$, *and* $X(t)$ *follows* $dX = aX\,dt + bX\,dW$. *Find* $dY$ *where* $Y = f(X,t)$.

We calculate $f_t = 0, f_x = 1/x, f_{xx} = -1/x^2$, using Itô's formula,

$$\begin{aligned} dY &= \left( aX\frac{1}{X} - (bX)^2 \frac{1}{2}\frac{1}{X^2} \right) + bX\frac{1}{X}\,dW \\ &= \left( a - \frac{1}{2}b^2 \right) dt + b\,dW \end{aligned}$$

If $a$ and $b$ are constant, we are tempted to conclude

$$Y(t) = \left( a - \frac{1}{2}b^2 \right) t + b\,W(t)$$

which turns out to be correct. The real application of this result is that we have just solved the sde for $X$:

$$X(t) = e^{Y(t)} = X(0)e^{\left(a-\frac{1}{2}b^2\right)t+b\,dW(t)}$$