Chapter 1

Bond Pricing (continued)

How does the bond pricing illustrated here help investors in their investment decisions? This pricing formula can allow the investors to decide for themselves what the ”true value” of a coupon bond is, based on the current market information, and also can prepare the investors for the possible movements in interest rates. If the observed market price deviates too much from your own calculation, it may suggest some arbitrage opportunities. When two bonds with similar maturities but different coupons are offered with prices, how do we compare? With the yield calculation it becomes a rather simple comparison: we usually prefer the bond with a higher yield. Of course there are other considerations involved, such as the seniority of the bond, which usually has a priority in receiving payments.

Given that the bond market is driven by the interest rates movement, we would ask why the interest rates are so important to the economy. This is a major topic in macroeconomics, and it could be summarized in an over simplified version as follows. The economic activity in a country can be described by the income, or GNP (Gross National Product), which consists of consumption, investment, government purchases, and net export. As interest rates rises, people tend to consume less because it is more expensive to borrow, and there is an incentive to save. At the same time, it makes it more expensive for firms to borrow to buy new equipments so investment will also hurt. For export, high interest rates will attract foreign investors to buy dollars to deposit in the US, which increases demands for dollars therefore the dollar will be more expensive in relation to foreign currencies, that will make our products more expensive to foreign consumers, so export will hurt. Combining all the relations, a higher interest rate would usually slow down the economy. On the other hand, low interest rates are supposed to cause high inflation, although it didn’t seem to happen in recent years.

Who can decide on the interest rates? Nobody. They are mostly derived from the market, as we see from the open auction for US treasury bonds/bills. However, the Fed Reserve does set the short term interest rate and they can buy and sell bonds to manipulate the interest rates of other maturities, as they did in several quantitative easing experiments. The basic principle is that high demands for a bond with particular maturity can raise the price, which in turn lower the rate for that maturity.

Chapter 2

Brownian Motion

2.1 Random Walk

Previously we used a **discrete** random variable to describe the change of a stock price, and obviously this description comes with many limitations. In continuous time models, we aim at two extensions: the capability of reducing the time step to infinitesimally small dt, and allowing the range of the changes to cover a continuous interval. However, not every continuously distributed random variable serves this purpose. In particular, we will need this distribution to be *infinitely divisible* and to exhibit the required time scaling behavior. For these reasons, we turn our attention to Brownian motion, as it is the model that fits in with our requirements, albeit with other limitations like any mathematical models.

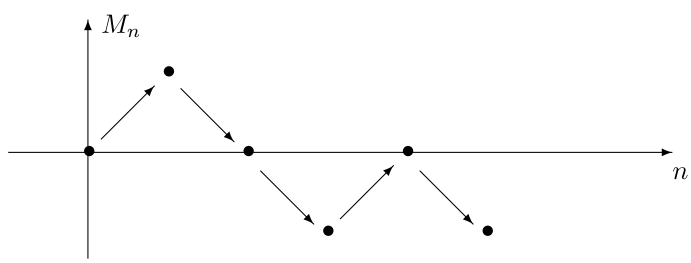
Recall that the stock price model we study in the binomial model can be expressed as

$$S_n = S_{n-1}u^{X_n} = S_0 u^{M_n}$$

where we introduce the random walk

$$M_n = \sum_{k=1}^{n} X_k, \qquad X_k = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

Here we assume the up move factor u and the down factor d satisfy $u \cdot d = 1$.



(In this plot, we have $X_1 = 1, X_2 = -1, X_3 = -1, X_4 = 1, X_5 = -1$.)

For illustration purpose, we let $p = 1/2$ (symmetric random walk) and consider the conditional probability

$$\mathbb{P}\left\{S_n = x \,|\, S_m = y\right\}, \quad m \leq n$$

which is the same as

$$\mathbb{P}\left\{M_n = \frac{\log(x/S_0)}{\log u} \,\middle|\, M_m = \frac{\log(y/S_0)}{\log u}\right\}, \quad m \leq n$$

We notice that the above will not be valid for arbitrary x and y, as M_n can only take integer values. To build a model for stock price paths, we need the joint probability distribution for the collection $\{M_1, M_2, \ldots\}$, which is rather easy to compute. This is one reason that we hope to extend the random walk by incorporating the time scaling and let the steps to go to zero.

Here are some of the most important properties of the symmetric random walk:

1. For $m \leq n$,

$$\mathbb{E}_m\left[M_n\right] = \mathbb{E}\left[M_n \,|\, \mathcal{F}_m\right] = M_m$$

2.

$$\mathbb{E}\left[M_n^2\right] < \infty, \quad \mathrm{Var}(M_n) = n, \quad \mathrm{Cov}(M_n, M_k) = n \wedge k$$

Proof:

1. We directly calculate

$$\mathbb{E}_m\left[M_n\right] = \mathbb{E}_m\left[M_m + X_{m+1} + \cdots + X_n\right] = M_m + \mathbb{E}_m\left[X_{m+1} + \cdots + X_n\right] = M_m$$

as $\mathbb{E}_m\left[X_k\right] = 0$ for $k > m$.

2. Since all X_k's are independent,

$$\mathrm{Var}(M_n) = \sum_{k=1}^{n} \mathrm{Var}(X_k) = n.$$

Assuming $n > k$,

$$\begin{aligned}
\mathrm{Cov}(M_n, M_k) &= \mathbb{E}[M_n \cdot M_k] - \mathbb{E}[M_n] \cdot \mathbb{E}[M_k] \\
&= \mathbb{E}\left[\mathbb{E}_k\left[M_n M_k\right]\right] \\
&= \mathbb{E}\left[M_k \cdot \mathbb{E}_k\left[M_n\right]\right] \\
&= \mathbb{E}\left[M_k \cdot M_k\right] \\
&= k
\end{aligned}$$

The variance and covariance results will imply that $\mathbb{E}\left[M_n^2\right] < \infty$.

2.2 Scaled random walk

Next we introduce some time scale to random walk and consider the following steps:
Step 1: Define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

for those t values such that nt is an integer. Based on the aforementioned properties, for s values such that $s \leq t$ and ns is also an integer,

(1).

$$\mathbb{E}_{ns}\left[W^{(n)}(t)\right] = W^{(n)}(s)$$

(2).

$$\mathrm{Var}(W^{(n)}(t)) = nt\left(\frac{1}{\sqrt{n}}\right)^2 = t, \quad \mathrm{Cov}(W^{(n)}(t), W^{(n)}(s)) = t \wedge s = s$$

Step 2: Now we fill the gaps for other t values by defining

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor}$$

where $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. This is a step function and it is discontinuous at those t such that nt is an integer.
Step 3: Finally we let n to go to infinity and the result is the standard Brownian motion

$$W(t) = \lim_{n\to\infty} W^{(n)}(t)$$

Why shall we scale it using a factor $1/\sqrt{n}$? Here are two reasons:

(A). We want the variance of $W^{(n)}(t)$ to scale like t.

(B). Furthermore, we need this so-called quadratic variation

$$\left[W^{(n)}, W^{(n)}\right](t) = \sum_{k}^{nt}\left[W^{(n)}\left(\frac{k}{n}\right) - W^{(n)}\left(\frac{k-1}{n}\right)\right]^2 = \sum_{k}^{nt}\left[\frac{1}{\sqrt{n}}X_k\right]^2 = t$$

Here we introduce a formal definition of Brownian motion: A real-valued continuous-time process $\{W(t)\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard Brownian motion if it satisfies

(1). The process starts from the origin:

$$W(0) = 0$$

(2). The increments

$$W(t_1) - W(t_0), W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots$$

with $t_0 < t_1 < t_2 < t_3 < \cdots$, are jointly independent.

(3). The distribution of an increment is normal:

$$W(t) - W(s) \sim N(0, t - s).$$

(4). $W(t)$ is continuous in t, but nowhere differentiable.

(5). The covariance

$$\mathrm{Cov}(W(s), W(t)) = s \wedge t$$

To verify a process to be a Brownian motion, we need to check all of the conditions above. For example, consider

$$X(t) = tW\left(\frac{1}{t}\right),\ t > 0$$

with $X(0)$ to be specified as 0. We proceed to note that (1) $X(0) = 0$ is already satisfied. For (2), we need to notice that if there is no overlap between intervals (t_1, t_2) and (t_0, t_1), there will be no overlap between $(1/t_2, 1/t_1)$ and $(1/t_1, 1/t_0)$, as the function $1/t$ is monotone. For (3), let's assume $s < t$ so $1/t < 1/s$, we first note

$$\begin{aligned}
&tW(1/t) - sW(1/s) \\
=&tW(1/t) - sW(1/t) + sW(1/t) - sW(1/s) \\
=&(t - s)W(1/t) + s\left(W(1/t) - sW(1/s)\right)
\end{aligned}$$

then we note $0 < 1/t < 1/s$ so $(0, 1/t)$ and $(1/t, 1/s)$ have no overlap.

$$\mathrm{Var}\left[tW(1/t) - sW(1/s)\right] = (t - s)^2 \cdot \frac{1}{t} + s^2\left(\frac{1}{s} - \frac{1}{t}\right) = t - s$$

For (4), we just need to note that for $t > s > 0$, we can use the above rearrangement to show that

$$(t - s)W(1/t) \to 0, \quad s\left(W(1/t) - sW(1/s)\right) \to 0$$

as $t \to s$. The calculation of the covariance for (5) is similiar.

Brownian motions are also the most natural example for martingales:

$$\mathbb{E}_s\left[W(t)\right] = W(s) + \mathbb{E}_s\left[W(t) - W(s)\right] = W(s)$$

We can also generate other martingales from the standard Brownian motion such as the following:

(1).

$$W^2(t) - t$$

(2).

$$e^{\alpha W(t) - \frac{\alpha^2 t}{2}}$$