Chapter 0

Course Overview

- Part A: Derivative Pricing

  We will lead discussions on financial derivatives based on the factor/source of uncertainty:

  1. Stock price - equity derivatives;
  2. interest rates - fixed income derivatives;
  3. credit events - credit derivatives;
  4. others, such as volatility level, exchange rates, energy level, etc.

  The mathematical descriptions of the factors can be summarized as ways to model the change over a short time period $Z_j = X_j - X_{j-1}$:

  1. Use of coin toss (in the binomial model):

  $$Z_j = \begin{cases} 1 & \text{if } \omega = H \\ -1 & \text{if } \omega = T \end{cases}$$

  2. Use of normal random variables

  $$Z_j \sim N(\mu, \sigma^2)$$

  for some specified parameter μ and σ. Once we add the time factor and let $\Delta t \to 0$, with several more requirements (such as independent increments), it will lead to a model for changes over infinitesimal dt

  $$dX = dW$$

  and $W(t)$ is the so-called Brownian motion.

- Part B: Trading Strategies

1. Portfolio theory;
2. Factor analysis and PCA (Principal Component Analysis);
3. Statistical arbitrage;
4. Alpha trading.

- Projects: we will work on four projects with the following themes:

  1. Bonding pricing;
  2. Option pricing;
  3. Optimal portfolio/simumlation;
  4. Alpha trading.

- Online Market Information: we recommend the following web sites for market information.

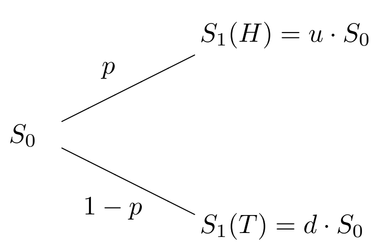
  1. Bloomberg
  2. Yahoo finance (for option quotes)

  We will be mostly interested in market information on

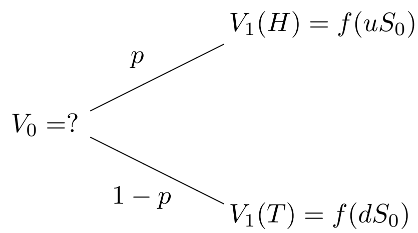
  - stock indices
  - individual stock prices
  - various interest rates
  - commodity prices
  - exchange rates
  - credit ratings of companies/municipalities
  - energy consumption levels

0.1 A quick summary of Math 5760

The basic one-period model for a stock price S, with parameters u, d and p, is described by the following diagram:



If a derivative with payoff function $f(S_1)$ is introduced, we have the following situations for the derivative:



V_0 is obtained as the no-arbitrage price so that a portfolio consisting of the stock and the derivative can be set up to eliminate the risk:

$$V_0 = \tilde{E}[f(S_1)] = \tilde{p}f(uS_0) + (1-\tilde{p})f(dS_0) = \frac{(1-d)f(uS_0) + (u-1)f(dS_0)}{u-d}$$

Here we assume that the risk-free interest $r = 0$. The probabilities $\tilde{p}$ and $1-\tilde{p}$ are called the risk-neutral probabilities and they are not related to the subjective probabilities p and $1-p$.

The one-period model can be extended to a multi-period model, and the time n no-arbitrage price of a European derivative that expires at time N is given by a conditional expectation

$$V_n = \tilde{E}_n[f(S_N)]$$

It is readily observed that this price process is a martingale in the risk-neutral probability measure.

In case of an American option, an optimal stopping time is introduced to give the optimal exercise, which will result in additional value in the case of an American put option.

There are several financial concepts that are essential to our discussions and we repeat them here:

1. A portfolio is a collection of financial assets with its price written as

   $$P(t) = \sum_{i=1}^{N} w_i(t) V_i(t)$$

   where we assume N assets with prices $V_i, i = 1, 2, \ldots, N$, the holdings are specified by numbers of shares $w_i, i = 1, 2, \ldots, N$. The changes in V_i are due to market conditions, and the changes in holdings are decided by the investors. It is noted that **short** positions ($w_i < 0$) are often allowed.

2. In financial investments, gain or loss are measured by the **return** over certain time period $[0, T]$:

   $$R = \frac{\Delta P}{P(0)} = \frac{P(T) - P(0)}{P(0)}$$

At $t = 0$, R can be only viewed as a random variable, therefore it is important to study the expected growth $E[R]$ and the risk level $Var(R)$. It is customary to model the return over $[t, t + \Delta t]$ as the sum of a deterministic and a random components:

$$\frac{\Delta P}{P} = \mu \Delta t + \sigma \epsilon$$

We usually introduce parameters μ and σ so that

$$\mathrm{E}\left[\frac{\Delta P}{P}\right] = \mu \Delta t, \quad \mathrm{Var}\left[\frac{\Delta P}{P}\right] = \sigma^2 \Delta t$$

and we have the following choices for ϵ:

(a) binomial model:

$$\epsilon = \begin{cases} \sqrt{\Delta t} & \text{if } \omega = H \\ -\sqrt{\Delta t} & \text{if } \omega = T \end{cases}$$

(b) continuous model:

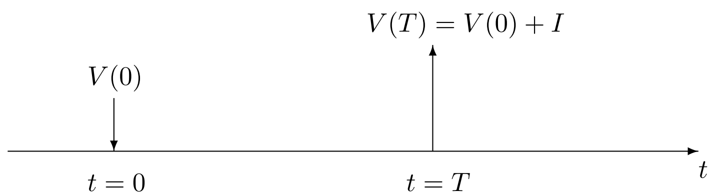
$$\epsilon \sim N(0, \Delta t)$$

Chapter 1

Bond Pricing

1.1 Compounding and present value

Imagine that you deposit an amount $V(0)$ in a bank, you would expect that you receive an amount $V(T) > V(0)$ when you withdraw at a later time T. The interest amount I will be proportional to the interest rate r, but the exact amount depends on how the interest is compounded. Assume $T > 0$ is an integer and $r > 0$ is the quoted annualized interest rate. The simple interest rate amount is just $I = rTV(0)$, but once compounding is introduced, we have the following:



(a). annual compounding:

$$I = V(0)(1+r)^T - V(0)$$

(b). monthly compounding:

$$I = V(0)\left(1 + \frac{r}{12}\right)^{12T} - V(0)$$

(c). daily compounding:

$$I = V(0)\left(1 + \frac{r}{365}\right)^{365T} - V(0)$$

(d). continous compounding:

$$I = V(0)e^{rT} - V(0)$$

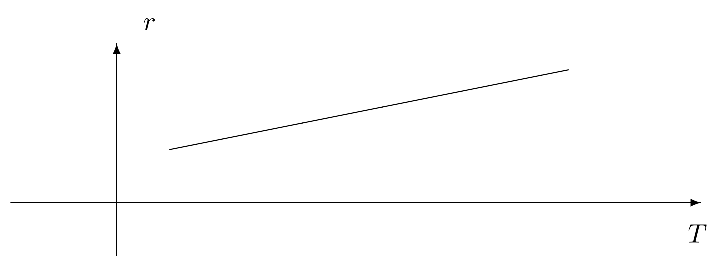
We will assume continuous compounding unless otherwise specified.

Now we discuss the concept of present value. Suppose there is a future cash flow C_T occurring at time $T > 0$, what is its current worth? Probably a more proper question to ask is the following: at what value C_0 the investors are *indifferent* to getting paid C_0 now at $t = 0$, or C_T at $t = T$? The answer becomes clear once we realize that the investor can deposit C_0 in a risk-free account earning interests, and receive $C_0 e^{rT}$ at T, which shows

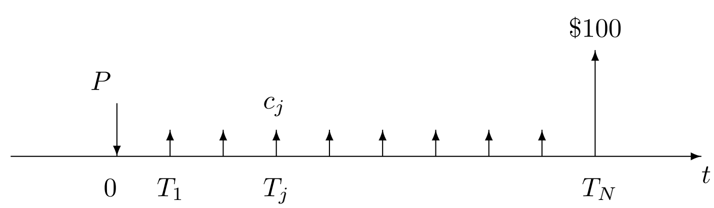
$$C_0 = C_T e^{-rT},$$

where r is the *risk-free* interest rate applied to the period $[0, T]$. It should be pointed out that this interest rate depends on the maturity T.

Where do we find such an interest rate? The zero-coupon bonds are introduced to address this need for market information. A zero-coupon bond is a bond with no coupon payments at all so there is just this single payment at the maturity. No such bond exists in practice but the professionals have come up with market prices for them. From these market prices we can infer the interest rates applied to these particular time periods. If there is enough information about rates for different maturities, we can describe such a maturity dependent interest rate as a function of T, this is the so-called interest rate *term structure.*



With an available interest rate term structure $r(T)$, we can strip a coupon bond into a collection of the following cash flows:



Here c_j is the coupon payment at time T_j. For example, if the face value (principal) of the bond is $100, with a coupon rate 2% and semiannual payments, we have $c_j = \$1$.

In general a coupon bond can be priced as

$$P = \sum_{j=1}^{N} c_j e^{-r(T_j)T_j} + 100\, e^{-r(T_N)T_N}$$

With the knowledge of a term structure, anyone can price a bond according to the above formula, and use it to compare with the price quoted from the market. If there is any disagreement, it should only be attributed to the term structure used. On the other hand, once the market price is observed, we can solve for the yield y in the equation

$$\sum_{j=1}^{N} c_j e^{-y \cdot T_j} + 100\, e^{-y \cdot T_N} = P_{market}$$

The *yield* of the coupon bond can be viewed as an average of the term structure $r(T)$, with more weight towards $r(T_N)$. It is this quantity an investor mainly looks at when he/she considers a coupon bond as an investment opportunity.