A Note on Parameter Estimation

This note explains the basic procedure to estimate parameters for the models involved in the final project. The model that relates the returns of the portfolio to the returns of the index fund is

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \beta \frac{dI(t)}{I(t)} + dX(t) \tag{1}$$

which is discretized into

$$R_n^S = \beta_0 + \beta R_n^I + \epsilon_n, \quad n = 1, 2, \ldots, 60. \tag{2}$$

Here we use daily returns $\{(R_1^I, R_1^S), (R_2^I, R_2^S), \ldots, (R_{60}^I, R_{60}^S)\}$ from the previous 60 days. The least square fit is to seek β_0, β to minimize

$$\sum_{n=1}^{60} \left(R_n^S - \beta_0 - \beta R_n^I\right)^2$$

The parameters β_0 and β are just the y-intercept and the slope of the line that best fits the data points (R_n^I, R_n^S). They are found by solving a 2×2 linear system that is obtained by taking the derivatives of the above function with respect to β_0 and β, and set both derivatives to zero. The solutions to the linear system are found to be

$$\beta_0 = \frac{CD - AB}{60D - A^2}, \quad \beta = \frac{60B - AC}{60D - A^2}$$

where

$$A = \sum_{n=1}^{60} R_n^I,\ B = \sum_{n=1}^{60} (R_n^I \cdot R_n^S),\ C = \sum_{n=1}^{60} R_n^S,\ D = \sum_{n=1}^{60} (R_n^I)^2$$

Once we have β_0, we can obtain

$$\alpha = \frac{\beta_0}{\Delta t} = 252 \cdot \beta_0$$

Now we can generate the noise in Eq.(2) as

$$\epsilon_n = R_n^S - (\beta_0 + \beta R_n^I), \quad n = 1, 2, \ldots, 60$$

and we assume

$$\epsilon_{n+1} = X_{n+1} - X_n, \quad n = 0, 1, 2, \ldots, 59$$

is the discretization for the mean reversion model for $X(t)$:

$$dX(t) = \kappa(m - X(t))\, dt + \sigma\, dW(t)$$

The solution to the above SDE is

$$X(t + \Delta t) = e^{-\kappa \Delta t} X(t) + (1 - e^{-\kappa \Delta t}) m + \sigma \int_t^{t+\Delta t} e^{-\kappa(t + \Delta t - s)}\, dW(s) \tag{3}$$

which can be written as

$$X_{n+1} = a + b\,X_n + \eta_{n+1}, \quad n = 0, 2, ..., 59.$$

We will assume $X_0 = 0$ for simplicity. The model calibration question is therefore to find parameters a and b such that the above would best fit the data set

$$\{X_1 = \epsilon_1, X_2 - X_1 = \epsilon_1, \ldots, X_{60} - X_{59} = \epsilon_{59}\}$$

or

$$\left\{X_0 = 0, X_1 = \epsilon_1, X_2 = \epsilon_1 + \epsilon_2, \ldots, X_{60} = \sum_{n=1}^{60} \epsilon_n\right\}$$

If we follow the similar least square approach, the problem to solve is

$$\min_{a,b} \sum_{n=0}^{59} \left(X_{n+1} - a - b\,X_n\right)^2$$

and the solutions are

$$a = \frac{GH - EF}{60H - E^2}, \quad b = \frac{60F - EG}{60H - E^2}$$

where

$$E = \sum_{n=0}^{59} X_n, \;\; F = \sum_{n=0}^{59} (X_n \cdot X_{n+1}), \;\; G = \sum_{n=0}^{59} X_{n+1}, \;\; H = \sum_{n=0}^{59} X_n^2$$

Once we have a and b, we can estimate

$$\mathrm{Var}(\eta) = \frac{1}{59} \sum_{n=0}^{59} \left(X_{n+1} - a - b\,X_n\right)^2$$

and κ, m and σ can be obtained as follows.

$$\begin{aligned}
\kappa &= -252 \log b \\
m &= \frac{a}{1 - b} \\
\sigma &= \sqrt{\frac{\mathrm{Var}(\eta) \cdot 2\kappa}{1 - b^2}}
\end{aligned}$$