Project No. 2, Math 5765, Due March 10 at 5 pm

1. On Feb 21, Bloomberg published a story: "The Next Financial Crisis Might be in Your Driveway". There is a warning that a new subprime bubble is on the verge of burst and this time it is the auto loan business, with its total debt reaching a peak of \$1.16 trillion at the end of 2016, according to the Federal Reserve Bank of New York. For this part of the project, we will explore the auto loan and auto lease markets, particularly about the relationship between a cash price and various payment plans. It should be pointed out that there are several ways of discounting: based on a) a zero-coupon bond price, and b) simple interest rate compounding, with the latter approach more familiar to the public, and that is what we assume here.

   (a) Consider the loan case first. If you are quoted a rate (annualized, but most likely with monthly compounding), then the discount takes the form $1/(1+r/12)^m$ where m is the time of the cash flow, measured in months. Suppose the loan amount is P, and there is a down payment D, and the term of the loan is for m months. The first payment is at the end of the first month, and the last payment is at the end of the term. Show that the monthly payment should be

   $$c = \frac{(P-D)r/12}{1-(1+r/12)^{-m}}$$

   (b) Now consider the lease case, in which the leasee pays a down payment at the beginning of the term, and monthly payments at every month end thereafter until the end of the term (T). At the end of the term, the leasee has an option to buy the car for the price of K, or just return the car with no other obligations. Let $S(t)$ be the market value of the car at time t. If the market value of the car at T is known, you can model the cash flow at T to be $\min\{S(T), K\}$, paid to the leaser (probably the dealer). List all the cash flows from both sides, and use the discount as in part (a), show that the monthly payment c should be satisfy

   $$S(0) - D = \frac{c}{1+r/12} + \frac{c}{(1+r/12)^2} + \cdots + \frac{c}{(1+r/12)^m} + \frac{\min\{S(T), K\}}{(1+r/12)^m}$$

   Solve for the monthly payment c.

   (c) Assume that the new car X has a market value \$25k, the interest rate for customer is quoted 3% and the lease term is 36 months. If you believe the car will be worth \$15k in three years, and the purchase option is specified at $K =$ \$15k, what monthly payment do you expect from this model?

   (d) In practice we never know the market value of the car in three years and some stochastic model is needed for $S(t)$. Let us use the Black-Scholes model

   $$\frac{dS}{S} = -\alpha\, dt + \sigma\, dW$$

Notice that the Brownian motion here is measured in the real world, and we use $-\alpha$ for $\alpha > 0$ to emphasize the depreciation. Make some comments on this choice as whether it is appropriate for our purposes. Would it be reasonable to use a constant α? What would you suggest for a time-dependent α?

(e) Assuming the price dynamics (d), extend (b) to derive a more realistic formula for c. If the customer has a probability 0.01 to default in the next year and this estimate of probability stays intact for the future, what extra interest would you charge the customer?

(f) Suppose car X has cash value \$30k and car Y has cash value \$25k, and both are advertised for 36-month lease. The monthly payment for X is offered at \$250 and for Y it is \$300. Use this model to give several numerical examples to explain this counterintuitive pricing. In particular, make some suggestions as what K the dealer could use to manipulate the offers.

2. We will derive the mortgage payment formula in a few steps. For a T-year (or $m = 12T$ months) mortgage, the lender offers a loan in an amount P to the borrower, and the borrower pays a monthly payment c each month (the first payment is at the end of the first month, and the last payment is at the end of the term). Let P_j to be the remaining principal right after the j-th payment, which covers the reduction of the remaining principal over that period, and the interest for the remaining principal over that period:

$$c = P_{j-1} - P_j + P_{j-1}\frac{r}{12} = P_{j-1}\left(1 + \frac{r}{12}\right) - P_j$$

(a) The above equation is a recurrence relation for $\{P_j, j = 0, \ldots, m\}$, with P_0 to be the original loan amount, and $P_m = 0$. Solve this recurrence relation and show that

$$c = \frac{P_0 a^m (a-1)}{a^m - 1}, \qquad a = \left(1 + \frac{r}{12}\right)$$

(b) All home mortgage loans in US have embedded options that allow the borrowers to prepay the principal at any time during the life of the mortgage. Describe this option and show that the option is on the prevailing mortgage rate, struck at the current rate the borrower signed up for when the mortgage was initiated. Discuss the implications of this embedded option and what the lender should do to compensate for the giveaway.