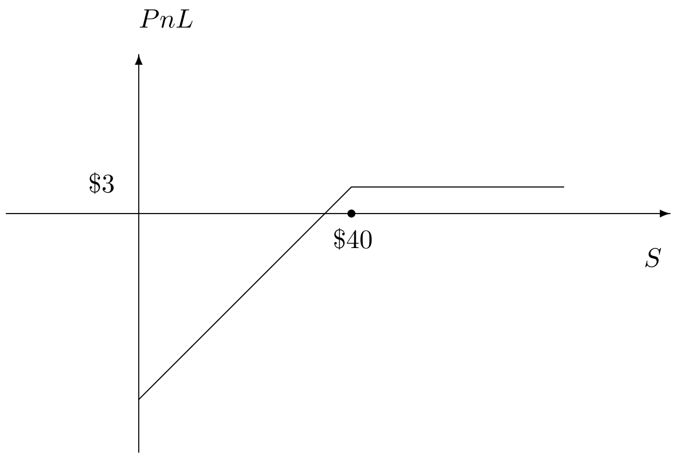
Notes for Homework Problems

Homework No. 1:

1. When you sell a put option, you are looking for the stock to stay above the strike. Ignoring the interest rate, the put option position will be exercised if the stock price falls below the strike price \$40. If it falls below \$37, the option loss would erase the \$3 the seller received at the sale of the option.



2. See the spreadsheet `5step_model.xlsx`.

3. See the spreadsheet `Bond_Pricer.xlsx`.

4. We just need to show that for $m \leq n$,

$$\mathbb{E}_m[X_n] = \mathbb{E}_m\left[X_m + \sum_{j=m+1}^{n} Z_j\right] = X_m$$

The crucial result is that

$$\mathbb{E}_m[X_j] = 0, \qquad \text{for } j > m$$

Homework No. 2:

1. (a) Notice that it's a piecewise constant function, with discontinuities at points where nt are integers.

   (b) We need to apply the central limit theorem to

$$W^{(n)}(t) = t \cdot \frac{X_1 + X_2 + \cdots + X_{\lfloor nt \rfloor}}{\sqrt{\lfloor nt \rfloor}}$$

(c) We can rewrite

$$W^{(n)}(t) = \left(W^{(n)}(t) - W^{(n)}(s)\right) + W^{(n)}(s)$$

and notice that these two parts are independent in the expectation calculations.

2. (a) To verify a process is a Brownian motion, we need to go through all the checks listed on pages 5-6 of the lecture notes for week 2. The main parts to check are the independent increments condition and the variance of $X(t) - X(s)$. Again, we should note

$$X(t) - X(s) = (W(T+t) - W(T)) - (W(T+s) - W(T)) = W(T+t) - W(T+s)$$

Since $W(t)$ is the standard Brownian motion, The mean of $W(T+t) - W(T+s)$ is zero, and the variance is the time elapsed between

$$(T+t) - (T+s) = t - s$$

(b) Similarly,

$$X(t) - X(s) = c\left(W\left(\frac{t}{c^2}\right) - W\left(\frac{t}{c^2}\right)\right)$$

and the variance is

$$c^2\left(\frac{t}{c^2} - \frac{s}{c^2}\right) = t - s$$

Homework No. 3:

1. We apply Itô's formula to $f(W) = \frac{1}{2}W^2$ to obtain

$$d\left(\frac{1}{2}W^2\right) = W\,dW + \frac{1}{2}(dW)^2 = W\,dW + \frac{1}{2}dt$$

2. We apply Itô's formula to $f(W) = e^{\sigma W}$ to obtain

$$dX = \sigma e^{\sigma W}\,dW + \frac{1}{2}\sigma^2 e^{\sigma W}\,(dW)^2 = \sigma e^{\sigma W}\,dW + \frac{1}{2}\sigma^2 e^{\sigma W}\,dt$$

Taking expectation, notice that the expectation of the first term is zero because of the dW factor,

$$dm(t) = \frac{1}{2}\sigma^2 m(t)\,dt, \quad \text{or } m'(t) = \frac{1}{2}\sigma^2 m(t), \quad m(0) = \mathbb{E}[e^{\sigma W(0)}] = 1$$

The solution of the ODE is

$$m(t) = e^{\frac{\sigma^2}{2}t}$$

3. We just need to show

$$\mathbb{E}[dY] = m\,dt, \quad \mathrm{Var}[dY] = \sigma^2 dt$$

or

$$\mathrm{Var}[\sigma_1 dW^{(1)} + \sigma_2 dW^{(2)}] = (\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)\,dt$$

Therefore

$$\sigma^2 = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

Homework No. 4:

1. We know the solution for S is

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma d\tilde{W}(t)}$$

(a)

$$\begin{aligned}
&\mathbb{P}\left\{K_1 < S(T) < K_2\right\} \\
=&\mathbb{P}\left\{K_1 < S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\tilde{W}(T)} < K_2\right\} \\
=&\mathbb{P}\left\{\frac{K_1}{S(0)}e^{-(r-\frac{1}{2}\sigma^2)T} < e^{\sigma\tilde{W}(T)} < \frac{K_2}{S(0)}e^{-(r-\frac{1}{2}\sigma^2)T}\right\} \\
=&\mathbb{P}\left\{\log\frac{K_1}{S(0)} - (r-\frac{1}{2}\sigma^2)T < \sigma\tilde{W}(T) < \log\frac{K_2}{S(0)} - (r-\frac{1}{2}\sigma^2)T\right\} \\
=&\mathbb{P}\left\{\frac{\log\frac{K_1}{S(0)} - (r-\frac{1}{2}\sigma^2)T}{\sigma} < \frac{\tilde{W}(T)}{\sqrt{T}} < \frac{\log\frac{K_2}{S(0)} - (r-\frac{1}{2}\sigma^2)T}{\sigma}\right\} \\
=&N(a) - N(b)
\end{aligned}$$

where $a = \frac{\log\frac{K_2}{S(0)} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $b = \frac{\log\frac{K_1}{S(0)} - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, and we note that $W(T)/\sqrt{T}$ is a standard normal.

(b) Note that we can represent the event as

$$\frac{S(T_2)}{S(T_1)} > 1, \quad \text{and} \quad \frac{S(T_1)}{S(t)} > 1$$

But these two are independent, so

$$\mathbb{P}\left\{S(T_2) > S(T_1) > S(t)\right\} = \mathbb{P}\left\{S(T_2) > S(T_1)\right\} \cdot \mathbb{P}\left\{S(T_1) > S(t)\right\}$$

Using the solution for S,

$$\frac{S(T_1)}{S(t)} = e^{(r-\frac{1}{2}\sigma^2)(T_1-t)+\sigma(W(T_1)-W(t))}$$

so

$$
\begin{aligned}
&\mathbb{P}\left\{\frac{S(T_1)}{S(t)} > 1\right\} \\
=&\mathbb{P}\left\{e^{(r-\frac{1}{2}\sigma^2)(T_1-t)+\sigma(W(T_1)-W(t))} > 1\right\} \\
=&\mathbb{P}\left\{(r-\frac{1}{2}\sigma^2)(T_1-t)+\sigma(W(T_1)-W(t)) > 0\right\} \\
=&\mathbb{P}\left\{\frac{W(T_1)-W(t)}{\sqrt{T_1-t}} > \frac{-(r-\frac{1}{2}\sigma^2)(T_1-t)}{\sigma\sqrt{T_1-t}}\right\} \\
=&1-N(-c) \\
=&N(c)
\end{aligned}
$$

where $c = \frac{(r-\frac{1}{2}\sigma^2)(T_1-t)}{\sigma\sqrt{T_1-t}}$.

2. The spread option payoff can be decomposed as a long position in the call option with strike K, and a short position in the call option with strike $K+D$. As $D \to 0$, these two positions will tend to cancel with each other; and when $D \to \infty$, the second position has almost no value so the spread option is reduced to the call with strike K.

Homework No. 5:

1. This is a calculation of expectation when the distribution of the random variable is normal, when we have
$$S(T) = S_0 e^{a+bZ}$$
where $a = (r-\frac{1}{2}\sigma^2)T$ and $b = \sigma\sqrt{T}$, and Z is a standard normal random variable. The expectation involved is
$$V_0 = e^{-rT}\mathbb{E}\left[\Lambda(S(T))\right]$$
   (a) Here $S^n = S_0^n e^{na+nbZ}$ and we use the useful formula that $E[e^X] = e^{E[X]+\frac{1}{2}\mathrm{Var}(X)}$ so
$$V_0 = e^{-rT}\sum_{n=1}^{N} a_n S_0^n \exp\left(na+\frac{1}{2}n^2b^2\right) = e^{-rT}\sum_{n=1}^{N} a_n S_0^n \exp\left(n\left(r+\frac{n-1}{2}\sigma^2\right)T\right)$$
   (b) Notice that the expectation of $\max(S(T)-K,0)$ leads to the Black-Scholes formula. Here we have instead $S^n = S_0^n e^{na+nbZ}$, which can be viewed as
$$S^n = \tilde{S}_0 e^{(\mu-\frac{1}{2}\tilde{\sigma}^2)T+\tilde{\sigma}W(T)}$$

where $\tilde{S}_0 = S_0^n$, $\tilde{\sigma} = n\sigma$, $\mu = n(r + \frac{n-1}{2}\sigma^2)$. We can borrow the result from the Black-Scholes formula to calculate

$$V_0 = e^{(n-1)(r+\frac{n}{2}\sigma^2)T} S_0^n N(\tilde{d}_1) - e^{-rT} K N(\tilde{d}_2)$$

where

$$\tilde{d}_1 = \frac{\log\left(\frac{S_0^n}{K}\right) + (\mu + \frac{1}{2}\tilde{\sigma}^2)T}{\tilde{\sigma}\sqrt{T}}, \quad \tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T}$$

2. The purpose of the problem is to balance the option portfolio so that it's both delta and vega neutral. The procedure is the following.

   (a) Calculate the vega of the first option with strike 50, say it's $vega_1$;

   (b) Calculate the vega of the second option with strike 60, say it's $vega_2$;

   (c) Short $10vega_1/vega_2$ contracts of the option with strike 60, now you have two options in the portfolio and the portfolio's vega is zero;

   (d) Calculate the delta of the portfolio: $10d_1 - 10\frac{vega_1}{vega_2}d_2$ where d_1 and d_2 are the deltas of the first and the second options;

   (e) Short $100 \times (10d_1 - 10\frac{vega_1}{vega_2}d_2)$ shares of the underlying stock;

   (f) Now you have a portfolio with the following assets

   $$1000C_1 - 1000\frac{vega_1}{vega_2}C_2 - 1000\left(d_1 - \frac{vega_1}{vega_2}d_2\right)S$$

   and the portfolio is both delta and vega neutral.

Homework No. 6:

1. This is a standard application of the separation of variables technique. It should be noted that in order to have

   $$P(T, r; T) = A(T, T)e^{-B(T,T)r} = 1$$

   for any r values, we must have $A(T, T) = 1$ and $B(T, T) = 0$. They serve as the terminal conditions so crucial to the success of the solutions.

2. In practice it's the combination of two approaches that works the best. Namely, you choose 3 points to have some basic ideas about the parameters, then use a least square fit to fine tune the parameters.

Homework No. 7:

1. In this case it is crucial to perform the correct calculation for the expectation. Case 1 is the independence case where price moves and pollution levels are independent. Case 2 is a more realistic case where the price moves and pollution levels are positively correlated. For our model, the final decisions are the same.

Homework No. 8:

1. This is explained in the notes for week 10.

2. This is straightforward calculation. The purpose is to show that for many utility functions

$$\mathbb{E}[u(X)] < u(\mathbb{E}[X])$$

   when $u'' < 0$.

3. (a) For each portfolio we can associate a point (σ_P, μ_P), connect it with the risk-free portfolio point $(0, r)$. Pick the line with the maximum slope (which will bound all other portfolio points), the portfolio that corresponds to this line (B in this example) is closest to the market portfolio.

   (b) This is line with slope 0.6667.

   (c) This can be obtained by looking at the slope of the line connecting the risk-free portfolio point to portfolio C, and it is 0.6058.

   (d) This point $(6.2\%, 6.5\%)$ sits above CML so it is not a realistic expectation for us. The most we can expect at this risk level is to combine portfolio B with the risk-free portfolio and the expected return will be

$$1.5 + 0.6667 \times 6.2 \approx 5.6335\%$$